INFINITE SETS

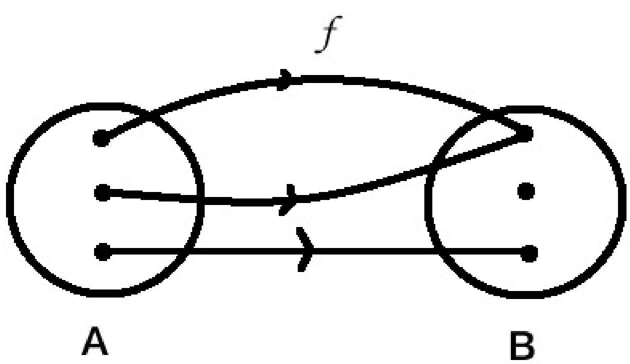
JUNIOR CIRCLE 10/23/2011

We have seen that

- If two finite sets A and B have the same number of elements, there is a function $f : A \to B$ that is both one-to-one and onto.
- If there is a function $f : A \to B$ that is both one-to-one and onto, the number of elements in A and B is the same.

We will call two such finite sets *equivalent*.

(1) Consider a function $f : A \to B$ defined by the diagram below



Is this function defined?

Is it onto?

Is it one-to-one?

Can you conclude that sets A and B have different numbers of elements?

(2) How can we compare the size of two *infinite* sets using functions?

Definition 1. Two sets A and B (finite or infinite) are *equivalent* if there is a function $f : A \to B$ such that

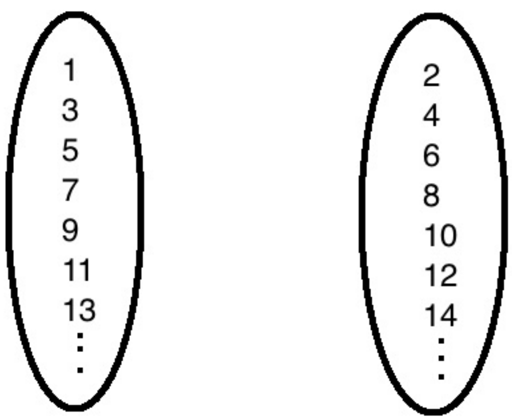
- f is one-to-one;
- f is onto;

Note that if we want to show that A and B are not equivalent, we have to prove that no such function exists.

Definition 2. A set A is called *countable* if it is equivalent to the set $\mathbb{N} = \{1, 2, 3, 4...\}$.

(1) Explain the name *countable*.

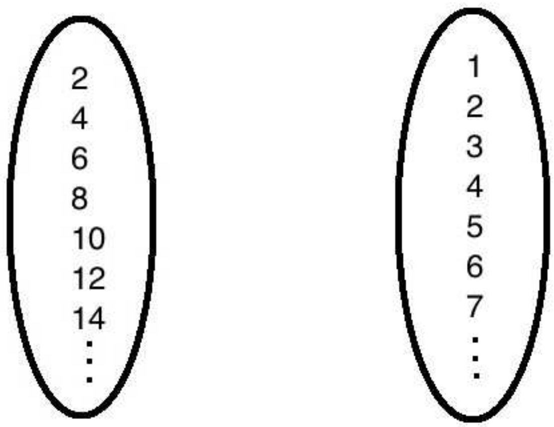
(2) Define a function from the set of odd numbers to the set of even numbers that is both $1-1$ and onto.



$$f(x) =$$

Are the sets of odd and even numbers equivalent?

(3) Can you define a function which is $1-1$ and onto from the set of natural numbers to the set of even numbers?



$$f(x) =$$

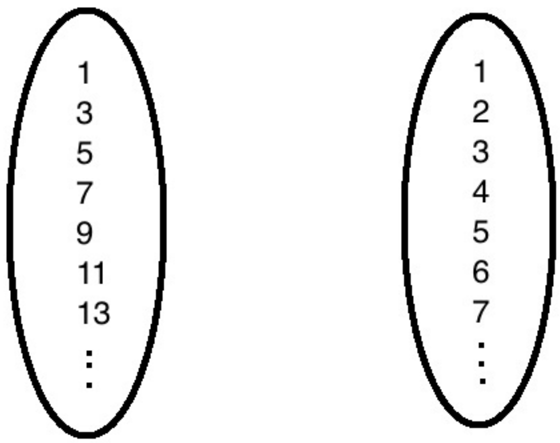
What does this mean about the set of even numbers?

(a) What is the corresponding function from the set of even numbers to the set of natural numbers that is $1-1$ and onto? (Reverse the function you found above)

$$f(x) =$$

Notice that this tells you how to number the elements of the set of even numbers. What number in the set of even numbers will be numbered 2011 in this enumeration?

(4) Can you define a function which is $1-1$ and onto from the set of natural numbers to the set of odd numbers? What does this imply about the set of odd numbers?



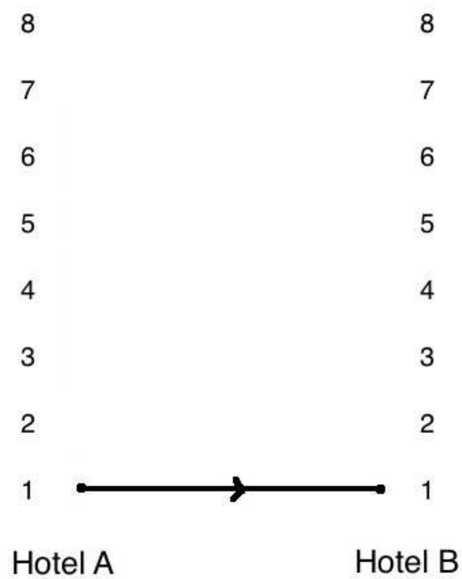
$$f(x) =$$

(a) What is the corresponding function from the set of odd numbers to the set of natural numbers that is $1-1$ and onto?

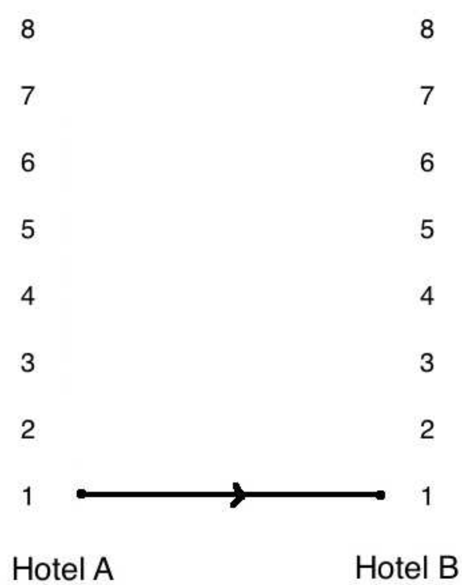
$$f(x) =$$

Notice that this tells you how to number the elements in the set of odd numbers. What number will be assigned to 2011 in this enumeration?

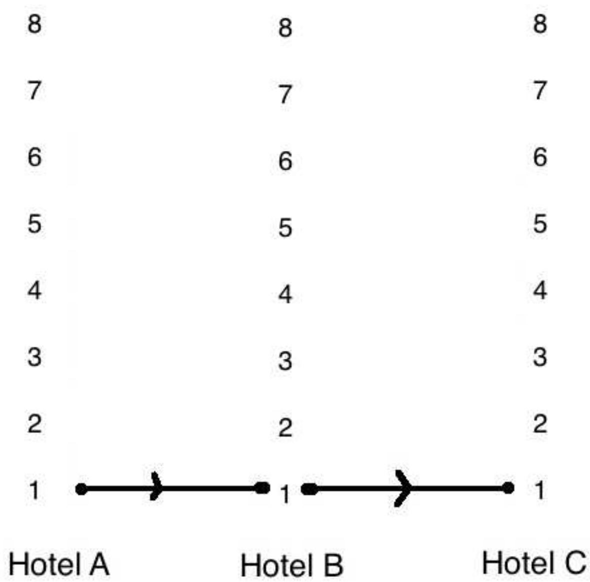
(5) Consider the set of rooms in two Hotels *Infinity*. Can you enumerate the elements of this set starting with Room 1 in Hotel A and continuing to Room 1 in Hotel B? *Hint:* This is how we distributed the keys when guests from two infinite hotels were moving to a third (empty) infinite hotel.



Can you find another way to do it?

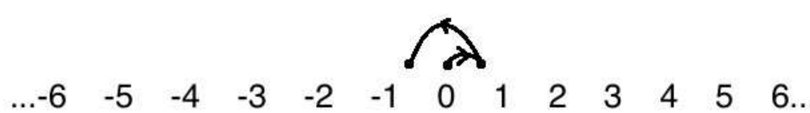


(6) Consider the set of rooms in three Infinity Hotels. Can you enumerate the elements of this set? *Hint:* This is how we distributed the keys when guests from three infinite hotels were moving to a fourth (empty) infinite hotel.



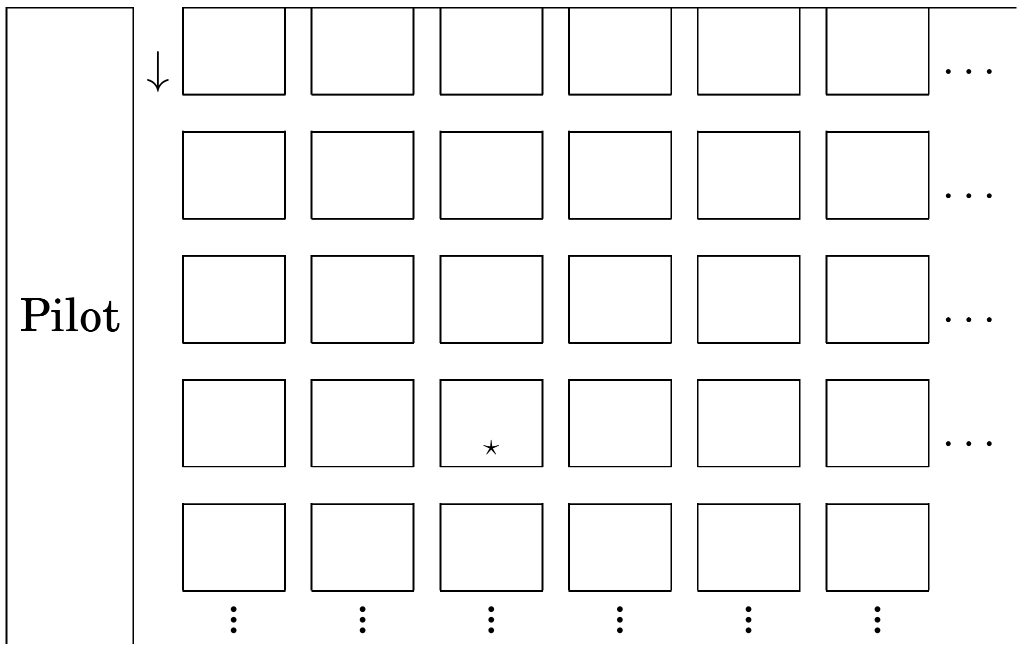
Does this mean that 3 Infinity Hotels are equivalent to 1 Hotel Infinity?

(7) Enumerate elements of $\mathbb{Z} = \{... - 3, -2, -1, 0, 1, 2, 3...\}$ starting in the following way $0 \mapsto 1 \mapsto -1 \ldots$.



Are the sets $\mathbb{Z}$ and $\mathbb{N}$ equivalent? Does a function $f : \mathbb{Z} \to \mathbb{N}$ which is both $1-1$ and onto exist? Can you define it?

(8) A super-wide-body rocket has an infinite number of seats in each row. Can you assign seats to an infinite number of passengers so that each seat is taken?



You can describe the position of any passenger in their seat by a pair of numbers. For example, the sit marked by $\star$ is described by the pair $(3, 4)$ (which means that you go 3 rows to the right and 4 rows down to get there from the entrance)

(a) Is the set consisting of pairs of natural numbers countable?

This means that infinitely many Infinity Hotels is equivalent to just one. AMAZING!