Geometry Homework Week 7

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1 Problems

1. (Trigonometric Ceva) AX, BY, CZ cevians of triangle ABC. Then, the mentioned cevians concur if and only if
$$\frac{\sin(\angle BAX)\sin(\angle CBY)\sin(\angle ACZ)}{\sin(\angle XAC)\sin(\angle YBA)\sin(\angle ZCB)} = 1.$$

2. (Mass Points?) In triangle ABC let AX,BY,CZ be cevians intersecting at P in the interior of $\triangle ABC$. Prove that $\frac{AP}{PX} = \frac{AZ}{ZB} + \frac{AY}{YC}$.

3. Let $\overline{AM}, \overline{BE}, \overline{CF}$ be concurrent cevians of a triangle ABC. Show that $\overline{EF}||\overline{BC}$ if and only if $BM = MC$.

4. Let M midpoint of BC in triangle ABC. Let G the centroid of ABC. Show that $\frac{AG}{GM} = 2$ by using negative homothety.

5. Prove that the figure formed when the midpoints of the sides of a quadrangle are joined in order is a parallelogram, and its area is half that of the quadrangle.

6. Prove that if one diagonal divides a quadrangle into two triangles of equal area, it bisects the other diagonal. Conversely, if one diagonal bisects the other, it bisects the area of the quadrangle.

7. Two circles are in contact internally at a point T. Let the chord AB of the larger circle be tangent to the smaller circle at a point P. Prove that the line TP bisects $\angle ATB$.

8. (Bashing Heron's Formula) In triangle ABC with $BC = a$, $CA = b$, $AB = c$, prove that
$$\sin^2(\angle A) = \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{4b^2c^2}$$
(Hint: Law of Cosines) Deduce Heron's formula
$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$
where $s = \frac{a+b+c}{2}$. (Hint: $\frac{1}{2}bc\sin A$)

9. Prove that if r is the inradius, s the semiperimeter, and R the circumradius, then
$$[ABC] = rs = \frac{abc}{4R} = \frac{1}{2}ab\sin C = 2R^2\sin A\sin B\sin C$$