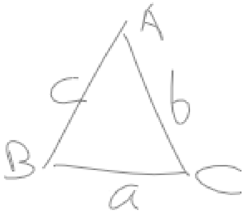
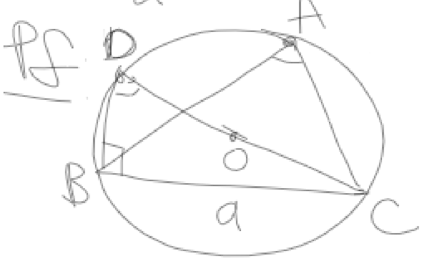
Thm: $\triangle ABC$, R

$$\frac{a}{\sin A} = 2R \left(= \frac{b}{\sin B} = \frac{c}{\sin C} \right)$$



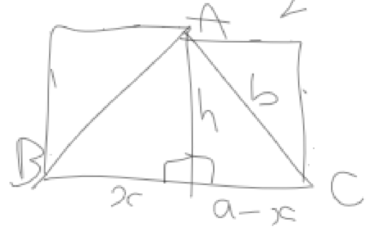
Pf:



$$\sin A = \sin D = \frac{a}{2R}$$

$\square$

Rk: $[ABC] = \frac{1}{2} ab \sin(C)$



$$[ABC] = \frac{1}{2} ha$$

$$[ABC] = \frac{abc}{4R}$$

P: $a = 13$, $b = 14$, $c = 15$. $R = ?$

Rk: $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$

$s = \frac{1}{2}(a+b+c)$

Prop: (Angle bisector thm)
ΔABC, $D \in [BC]$ s.t. AD bisects $\angle BAC$

$$\frac{AB}{AC} = \frac{DB}{DC}$$

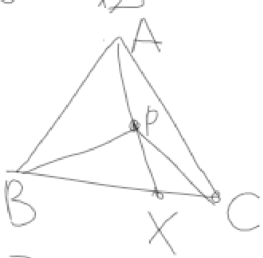
$$\parallel$$

Pf: $$\frac{\frac{1}{2} AB \cdot AD \cdot \sin\left(\frac{A}{2}\right)}{\frac{1}{2} AC \cdot AD \cdot \sin\left(\frac{A}{2}\right)} = \frac{[ABD]}{[ADC]}$$

$$= \frac{\frac{1}{2} BD \cdot AD \cdot \sin \angle BDA}{\frac{1}{2} DC \cdot AD \cdot \sin \angle ADC}$$

$$= \frac{BD}{DC} \cdot \boxed{\cancel{}}$$

Prop:



$$\frac{[ABP]}{[ACP]} = \frac{BX}{CX}$$

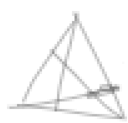
Rk: $\frac{a}{b} = \frac{x}{y} = \alpha \Rightarrow \alpha = \frac{a \pm x}{b \pm y}$

Pr

Pf: $$\frac{[ABX]}{[ACX]} = \frac{\frac{1}{2}|XB||AX| \sin\angle BXA}{\frac{1}{2} XC \cdot AX \cdot \sin\angle BXA} = \frac{XB}{XC}$$

$$\frac{[PBX]}{[PCX]} = \frac{[ABP]}{[ACP]}$$ $\boxtimes$

Thm: (Ceva)



Def:

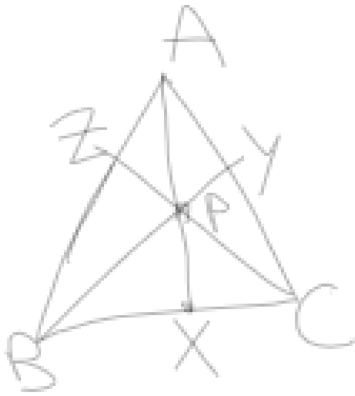
ΔABC, $X \in \overline{CB}$, $Y \in \overline{AC}$, $Z \in \overline{AB}$

[AX, BY, CZ concur at a pt]

$\Updownarrow$

$$\left[\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1\right]$$

Pf ("$\Longrightarrow$")



$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=\frac{[BPA]}{[APC]}\,\frac{[BPC]}{[BPA]}\,\frac{[APC]}{[BPC]}$$

$$=1$$

("$\Longleftarrow$") Phantom pts

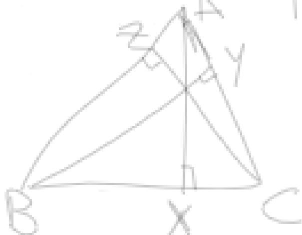
$$\frac{BX}{XC}\cdot\boxed{\;''\;}=1$$

$$\frac{BX'}{X'C}\cdot\boxed{\;''\;}=1$$

$$\frac{BX}{XC}=\frac{BX'}{X'C}\Rightarrow X=X' \quad \square$$

Ex: Show $\left\{\begin{array}{l}\text{orthocenter}\\ \text{incenter}\\ \text{centroid}\end{array}\right\}$ exist

Pf:



Def Directed lengths:

$$\frac{AZ}{ZB} = \pm \frac{|AZ|}{|ZB|} \quad \begin{cases} + & A, Z, B \\ - & Z \notin [AB] \end{cases}$$

Thm (Ceva, Directed)

$\triangle ABC$, X, Y, Z as above (Ceva)

X, Y, Z (distinct from vertices)

AX, BY, CZ concurrent $\Leftrightarrow \left(\frac{AZ}{ZB} \frac{BX}{XC} \frac{CY}{YA} = 1\right)$

[illegible]

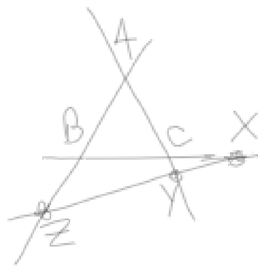
$\ldots \frac{}{YA}$!

(Menelaus thm.) (distinct from vertices)

$\triangle ABC$, (X, Y, Z) on (BC, CA, AB)

X, Y, Z collinear $\Leftrightarrow \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$

Pf: $(\Leftarrow)$



p, q, r

; $\frac{q}{r} = \frac{BX}{XC}$, $\frac{r}{p} = \frac{CY}{YA}$ $\frac{p}{q} = \frac{AZ}{ZB}$

$$\left.\begin{aligned} AA_1 &= p \\ BB_1 &= q \\ CC_1 &= r \end{aligned}\right\}$$

$$\underbrace{\frac{CC_1}{BB_1}}_{\frac{q}{r}} = \frac{CX}{BX}$$

$\square$

In the next meeting there will be a poll with fruits. S

Very Strong recommendation: choose plum

Centroid

Ex1:

Pf: $x = x$

$2y = 2z$

$$\frac{\boxed{[BGA]}}{[CGA]} = \frac{BX}{CX} = 1$$

Ex2:

$$\frac{AG}{GM} = 2$$

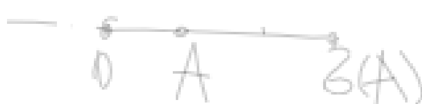
Homothety

$E \xrightarrow{\sigma} E$ $\quad A^k$

$\exists O \in E,\ k \in \mathbb{R}\setminus\{0\}$ s.t. $\forall A \in E$,

$O, A, \sigma(A)$ collinear

and $\frac{\sigma(A)O}{OA} = -k$



Lemma: ΔABC, ΔXYZ, $AB \parallel XY$,

$BC \parallel YZ$,

$CA \parallel XY$