Lesson 5: Quadratic Equations IV

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Problem 0.
Let \( f(x) = ax^2 + bx + c \) be a quadratic equation with \( a > 0 \).
Show that \( f \) achieves its unique minimal value at \( -b/(2a) \). In other words, show that for any \( x \neq -b/(2a) \) we have
\[
f(x) > f\left(\frac{-b}{2a}\right)
\]
Show that if \( a < 0 \), then similarly \( f \) achieves its unique maximal value at \( -b/(2a) \).

Problem 1.
Show that the equation \( x^2 + px - 1 \) has two distinct real roots for all values of \( p \).

Problem 2.
a) Find a quadratic equation with roots \( \sqrt{2} \) and \( -\sqrt{7} \). Is it unique?
b) Find a quadratic equation with integer coefficients and a root \( 4 - \sqrt{7} \).

Problem 3.
a) Two real roots of the equation \( ax^2 + bx + c = 0 \) have difference 2020. What is the discriminant of this equation if \( a = 1 \)?
b) Prove that the equation \( ax^2 + 2bx + 4c \) has two roots.
c) What is the difference between them?

Problem 4.
Is it true that if \( b > a + c > 0 \), then the quadratic equation \( ax^2 + bx + c = 0 \) has two distinct real roots?

\[ \text{Hint: Look at } f(-1) \text{ and } f(1). \]

Problem 5.
All three coefficients of a quadratic equation are odd integers. Show that it cannot have a root of the form \( 1/n \), where \( n \) is an integer.