2. Induction

One of the most powerful techniques in mathematics is the method of induction.
Induction works on the following principle: Let $P(n)$ be a logical statement that depends on the number n.
Suppose that we know that $P(1)$ is true.
Also, suppose that whenever $P(k)$ is true, then $P(k+1)$ is also true
.
Then it is the case that the statement $P(n)$ is true for every n.
Probably the easiest way to learn about induction is to see an example. Here is a very simple induction problem

Problem 9. Morgan likes pizza a lot. Morgan can always eat another slice of pizza– that is, if Morgan can eat a pizza with 5 slices, he has no problem with eating a pizza with 6 slices. Additionally, we know that Morgan has no problem eating a single slice of pizza. Show that Morgan can eat pizzas of arbritary size.

The statement $P(n)$ is then

$$P(n) = \text{ Morgan can eat Pizzas with } n \text{ slices}$$

We know additionally that

$$P(1) = \text{ Morgan can eat pizzas with 1 slice}$$

is also true, and additionally

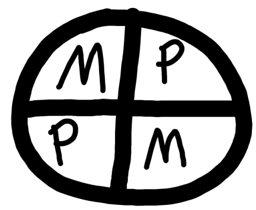
$$P(k) \Rightarrow P(k{+}1) = \text{ If Morgan can eat a pizza with } k \text{ slices, he can eat one with } k+1 \text{ slices}$$

is also true.
It logically follows by the principle of induction, that Morgan can eat a pizza with any number of slices; that is the statement $P(n)$ is true for every n. Let us look at a more

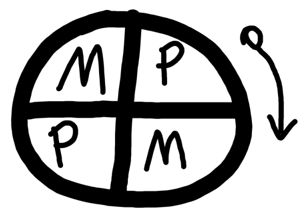
difficult problem.

Problem 10. Derek is about to eat a pizza. The pizza has an even number of slices, and half the slices have pepperoni, and the other half the slices have mushrooms. The are randomly assorted. Derek likes pepperoni, and dislikes mushrooms. The way Derek eats pizza is like this: He starts by eating a slice of pizza, and then eats the slice clockwise of that one, and then the slice clockwise of that one, and so on until the pizza is finished. Derek is ok with eating a slice with mushrooms, as long as at any given point he has eaten at least as many slices of pepperoni as slices with mushrooms. Why is it that Derek can happily eat his pizza?

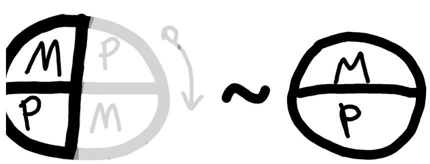
First, let's look at an example. Here is a pizza with four slices.



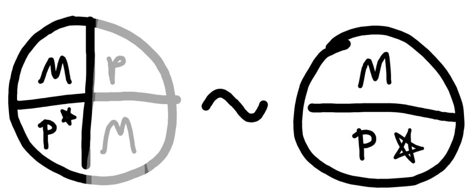
If Derek starts at the marked piece, then he will be able to eat his pizza happily in a clockwise direction.



Let's now prove that every pizza with 4 slices can be enjoyed by Derek.
Fact: There always is a mushroom slice that lies clockwise and next to a pepperoni slice.
If we remove these two adjacent slices, we get something that looks like a pizza with 2 slices, and clearly every pizza with 2 slices can be enjoyed by Derek.



Let's mark the slice where Derek must start in order to enjoy his 2-slice pizza



Now what happens if we add the two slices back in? Because the mushroom slice lies clockwise of the pepperoni slice, Derek will eat the pepperoni slice first and then the mushroom slice. Derek can still enjoy his pizza with the two slices back in.
Can we extend this process to bigger and bigger pizzas? Let us fit this in the language of induction:

Problem 11. To solve this problem, we need to phrase it in a logical statement that depends on a number n.

(i) What is the statement $P(n)$ that we are trying to prove. Write out your statement in a complete statement.

(ii) The next part of induction is to show that the initial statement, $P(1)$ is true. Write the initial statement $P(1)$ in full sentences, and explain why it is true.

(iii) The next part of induction is to prove the "inductive step" that is
if $P(k)$ is true, then $P(k+1)$ is true
Can you write the explain what the above statement is in full sentences for this problem?

This is the tricky statement to prove, but it is the one that involves removing

the 2 special slices, and adding them back in.

(iv) Explain why

$P(k)$ then $P(k+1) =$ If Derek enjoys every pizza with $2k$ slices, then he can enjoy every pizza with $2k+1$ slices

is true in detailed sentences.

3. Additional Induction Problems

Problem 16. Show (using induction) that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

(i) Write out the statement $P(n)$?

(ii) What is $P(1)$ say, and why is it true?

(iii) Prove that if $P(n)$ is true, then $P(n+1)$ is true.