The Fundamental Theorem of Algebra

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Definition 1 *Recall that a complex number is a number of the form $z = a + bi$ where a and b are real numbers and $i^2 = -1$. Every complex number also has a polar form $z = re^{i\theta}$, where r is a nonnegative real number and $e^{i\theta} = \cos\theta + i\sin\theta$.*

Problem 1:

a. Convert the following complex numbers into polar form.

   i. $z = 1 + i$

   ii. $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

   iii. $z = -1$

b. Convert the following into $a + bi$ form.

   i. $z = e^{2\pi i/3}$

   ii. $z = 2e^{\pi i/2}$

   iii. $z = \sqrt{2}e^{3\pi i/4}$

Definition 2 *A (complex)* ***polynomial*** *is a function $p(z)$ of a complex variable z that can be written as*

$$p(z) = a_n z^n + ... + a_1 z + a_0$$

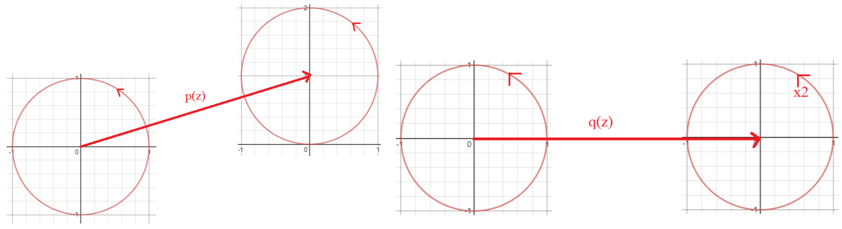
where $a_n, ..., a_1, a_0$ are complex numbers and $a_n \neq 0$. The largest power term, in this case n, is called the ***degree*** *of the polynomial.*

Our main goal is to prove the Fundamental Theorem of Algebra, in the following form

Theorem 1 *(Fundamental Theorem of Algebra) Every degree n polynomial has exactly n roots in the complex plane, counting multiplicities.*

One major difficulty that arises is that polynomials are functions of complex numbers which output complex numbers, and so are very difficult to graph, especially on a piece of paper. One way to graph complex polynomials is to draw a "nice" shape (such as a circle) in the domain and see what it is mapped to in the range.

Example 1 *The following are graphs of $p(z) = z + i$ and $q(z) = z^2$, where we have chosen the circle of radius 1 around the origin in the domain.*



It is important to keep track of which direction we are drawing the circle in. Conventionally, we will choose the circle in the domain to rotate once counterclockwise, but this may not necessarily be the case for the range, as we see in the second example in particular. Less importantly, it can be convenient (but not always necessary) to fix a starting point on the circle, which will also be our ending point.

Problem 2: To graph polynomials it will help to keep in mind the polar form of a complex number.

a. Verify that the graphs in Example 1 are correct.

b. Draw (or describe) the image of the circle of radius 1 about the origin for each polynomial below.

   i. $p(z) = z^n$ (n is a positive integer)

   ii. $p(z) = cz^n$ (c is a positive real number)

   iii. $p(z) = e^{i\theta} z^n$ (θ is a real number)

   iv. $p(z) = z^n + a + bi$ (a, b real numbers)

c. Describe the kinds of polynomials you can easily graph similarly to part (b).

Problem 3: We now start considering some harder polynomials.

a. Draw the image of the circle of radius 1 about the origin for $p(z) = z^2 - z + 1$ (Hint: Try graphing each term separately and "adding" together the graphs. It helps to have a starting point in mind.)

b. Now let's take a circle of radius $R = 5$ about the origin, or $R = 10$. Try to get a good idea of what's different. (Hint: Maybe you can slightly deform this picture into one you're familiar with.)

c. Now try $R = 0.2$ or $R = 0.1$. What's the difference now?

Problem 4: Now consider a general polynomial $p(z) = a_n z^n + ... + a_1 z + a_0$

a. Consider the image of the circle around the origin with radius R, where R is very large. Based on what we saw in Problem 3, what will this graph look like? It may also help to work out additional examples.

b. Consider the image of the circle around the origin with radius R, where R is very small. Based on what we saw in Problem 3, what will this graph look like?

c. (Challenge) Prove your answers for parts (a) and (b).

We've seen the images of circles centered at the origin, but what about circles centered away from the origin? To handle these it is convenient to convert our polynomials into a different form.

Theorem 2 *Every polynomial $p(z) = a_n z^n + ... + a_1 z + a_0$ can be rewritten as $p(z) = b_n(z - z_0)^n + ... + b_1(z - z_0) + b_0$ for any point z_0 in the complex plane.*

To handle a circle centered at z_0 it is usually a good idea to "re-center" our polynomial in this way.

Problem 5:

a. Re-center each polynomial below about the desired point z_0

   i. $z^2 - 2z + 1$ about $z_0 = 1$

   ii. $p(z) = z^3 - 3iz^2 - 3z + 3i$ about $z_0 = i$

b. Draw the image of the circle of radius 1 about z_0 for each polynomial below

   i. $z^2 - 2z + 1$ about $z_0 = 1$

   ii. $p(z) = z^3 - 3iz^2 - 3z + 3i$ about $z_0 = i$

Problem 6: Now suppose we have a circle centered at z_0 with radius R, where R is either very large or very small. What does its image under $b_n(z-z_0)^n + ... + b_1(z-z_0) + b_0$ resemble in either case? Compare your answers to Problem 4.

To prove the Fundamental Theorem of Algebra, one more result is needed, which we will state but not prove.

Theorem 3 *For every polynomial $p(z)$, there exists a z_0 in the complex plane that minimizes the value of $|p(z)|$.*

Problem 7: We're now ready to prove the Fundamental Theorem of Algebra.

a. What does it mean for z_0 to minimize the value of $|p(z)|$? What does it mean if $|p(z_0)| = 0$?

b. Now assume $p(z_0)$ is not zero. Consider circles of radius R around z_0, and make R very large and very small. Which case gives us some useful information? (Hint: Re-evaluate our assumption that $p(z_0)$ is not zero. Can this actually be the case? Why or why not?)

c. Complete the proof of the Fundamental Theorem of Algebra. (Hint: induction)

There are several ways to generalize the approach outlined by Problem 7. One such way results in a powerful theorem we will state, but not prove.

Theorem 4 *(Rouché's Theorem) Let Γ be the circle of radius R around the origin. If f and g are polynomials such that $|g(z)| < |f(z)|$ for all z lying on Γ, then f and $f+g$ have the same number of roots (counting multiplicities) lying inside of Γ.*[†]

Problem 8: (Challenge)

a. Prove that both roots of $p(z) = z^2 - \frac{1}{2}z + \frac{1}{4}$ have absolute value at most 1.

b. Prove that all five roots of $p(z) = z^5 + 3z^2 + 7$ have absolute value at most 2.

c. Using Rouché's Theorem, give another proof of the Fundamental Theorem of Algebra.

[†] In fact, Rouché's Theorem applies to a more general class of functions, known as *holomorphic functions*, as well as to a more general class of closed curves Γ. Such generality is beyond the scope of this worksheet, but the important fact is that all polynomials are holomorphic.