ORMC: PROBLEM SESSION

OLYMPIAD GROUP 1, WEEK 7

Problem 1. We are given a sequence of n boxes, together with $n-1$ comparison symbols in between them ($<$ or $>$). Show that one can arrange the numbers from 1 to n into the boxes such that the inequalities are respected. For example, for the sequence

$$\square < \square < \square > \square < \square < \square,$$

one could assign

$$\boxed{1} < \boxed{2} < \boxed{5} > \boxed{3} < \boxed{4} < \boxed{6}.$$

Problem 2. (Inspired by BAMO 8, 2016, C). Let $k \geqslant 2$ be an integer, let $A := 2^k - 2$ and $B := 2^k A$. Find the *smallest* nonnegative integer m such that

$$(A + m) \nmid (B + m).$$

Problem 3. (Inspired by BAMO 8, 2017, E). Let $\triangle ABC$ be an acute triangle with side lengths a, b, c. We call a set of rectangles $R_1, \dots, R_n$ a *good cover* of $\triangle ABC$ if:

(i). All rectangles have a common side length $h > 0$.
(ii). The rectangles and their interiors completely cover the interior of $\triangle ABC$ together.
(iii). The sum of the areas of the rectangles is exactly twice the area of $\triangle ABC$.

Do the following:

(a) Find a good cover of $\triangle ABC$ with n rectangles, for any $n \geqslant 1$.
(b) Find a good cover of $\triangle ABC$ with 3 rectangles of sizes $h \times a$, $h \times b$, $h \times c$ (for some h).

Problem 4. (Inspired by BAMO 8, 2016, E) A $2n \times 2n$ board is given.

(a) (Easy) Show that no matter how we pick $3n^2 + 1$ tiles, there will be four of them which are the vertices of a rectangle.

(b) (Harder) Show that no matter how we pick $2n(n + 1)$ tiles, there will be four of them which are the vertices of a rectangle.

Problem 5. (Inspired by BAMO 8, 2017, C). We are given a square $ABCD$ of area 256. Point E is inside the square, at equal distances from points C and D.

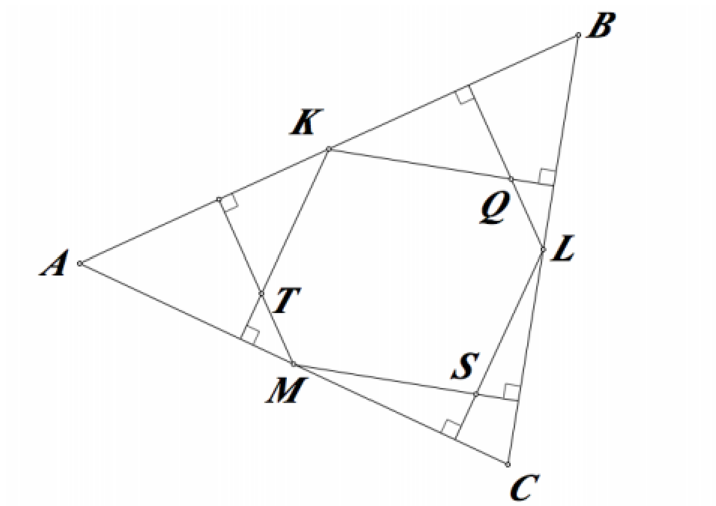
(a) If $\angle CED = 120°$, what is the area of $\triangle AEB$?
(b) If $\angle CED = 150°$, what is the area of $\triangle AEB$?

Problem 6. (Inspired by BAMO 8, 2018, C). Find all integers c such that the equation

$$m^2 + 18m + c = n^2$$

has infinitely many solutions in positive integers (m, n).

Problem 7. (BAMO 8, 2016, D). Let $\triangle ABC$ be an acute triangle, and let K, L, M be the midpoints of AB, BC and CA respectively. Draw perpendiculars from each of these midpoints to the other sides two of $\triangle ABC$; these intersect in points Q, S, T as shown below:



Show that the hexagon $KQLSMT$ has half of the area of the original triangle $\triangle ABC$.

Problem 8. (BAMO 8, 2014, C) Amy and Bob play a game. They alternate turns, with Amy going first. At the start of the game, there are 20 cookies on a red plate and 14 on a blue plate. A legal move consists of eating two cookies taken from one plate, or moving one cookie from the red plate to the blue plate (but never from the blue plate to the red plate). The last player to make a legal move wins; in other words, if it is your turn and you cannot make a legal move, you lose, and the other player has won.

Which player can guarantee that they win no matter what strategy their opponent chooses? Prove that your answer is correct.

There's no homework for next week; have a very happy Thanksgiving weekend and stay safe!