Quadratic Inequalities

Konstantin Miagkov

1 From Last Week

Problem 2.
A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if $f(x) = f(-x)$ for all $x \in \mathbb{R}$. Similarly, a function is called *odd* if $f(x) = -f(-x)$ for all x.
b) Show that any function from $\mathbb{R}$ to $\mathbb{R}$ can be uniquely written as a sum of an even and an odd function.

Problem 3.
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(2x + 1) = 4x^2 + 14x + 7$.

Problem 4.
Five integers are written on the board – three coefficients of a quadratic equation and two roots in arbitrary order. After one of the numbers is erased, the numbers $2, 3, 4, -5$ are left. What number was erased?

Problem 5.
Let $ABCD$ be a quadrilateral such that there exists a circle tangent to all of its four sides. Such a quadrilateral is called *circumscribed*. Show that $AB + CD = AD + BC$.

2 New Problems

Problem 1.
Let $f(x) = ax^2 + bx + c$ be a quadratic equation with $a > 0$.
a) Show that if f has no real roots, then $f(x) > 0$ for all real x.
Hint: complete the square!

b) Show that if f has exactly one real root x_0, then $f(x) > 0$ for all real $x \neq x_0$.

c) Show that if f has exactly two real roots $x_0 < x_1$, then $f(x) < 0$ for all real $x_0 < x < x_1$ and $f(x) > 0$ for all $x > x_1$ and $x < x_0$.

d) Formulate and prove the analogues of parts a), b), c) for the case when $a < 0$.

Problem 2.
Let $f(x) = ax^2 + bx + c$ be a quadratic equation with $a > 0$.
Show that f achieves its unique minimal value at $-b/(2a)$. In other words, show that for any $x \neq -b/(2a)$ we have

$$f(x) > f\left(\frac{-b}{2a}\right)$$

Show that if $a < 0$, then similarly f achieves its unique maximal value at $-b/(2a)$.

Problem 3.

Find all solutions to the equation $x(x+1) = 2018 \cdot 2019$