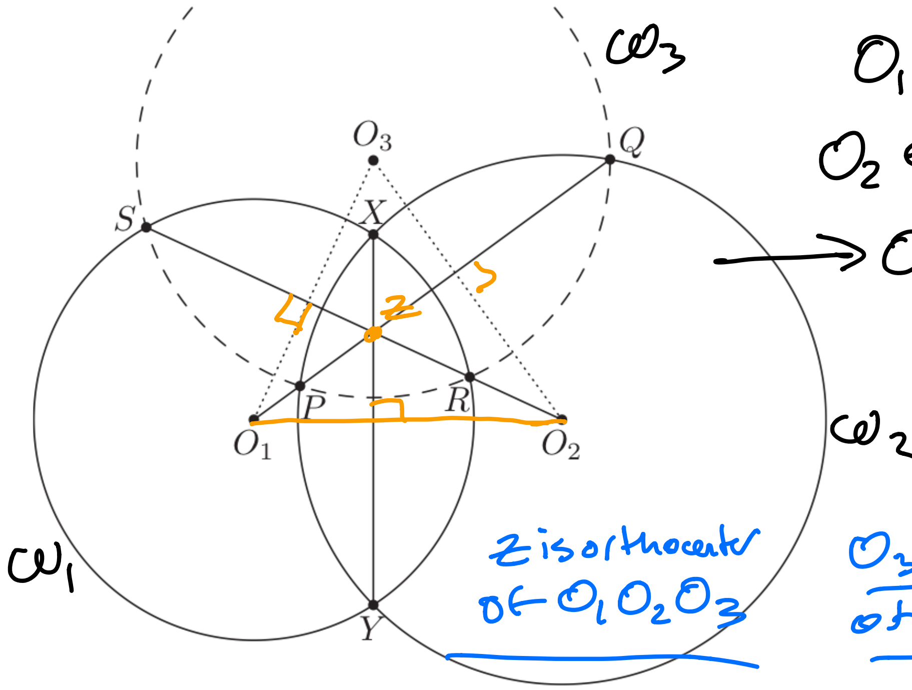
J **Example 2.21 (USAMO 2009/1).** Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R, and S lie on a circle then the center of this circle lies on line XY.



$O_1 \in RA(\omega_2, \omega_3)$

$O_2 \in RA(\omega_1, \omega_3)$

$\rightarrow O_3 \in RA(\omega_1, \omega_2)$

Z is orthocenter of $O_1O_2O_3$

O_3 orthocenter of O_1O_2Z

Figure 2.7A. The first problem of the 2009 USAMO.

$$\mathrm{Pow}_{\omega_2}(O_1) = \underline{O_1O_2^2 - r_2^2}$$
$$= \mathrm{Pow}_{\omega_3}(O_1) = \underline{O_1O_3^2 - r_3^2}$$
$$\mathrm{Pow}_{\omega_1}(O_2) = \underline{O_1O_2^2 - r_1^2}$$
$$= \mathrm{Pow}_{\omega_3}(O_2) = \underline{O_2O_3^2 - r_3^2}$$

WTS
$$O_1O_3^2 - r_1^2$$
$$= O_2O_3^2 - r_2^2$$
✗

Subtract equations:
$$r_1^2 - r_2^2 = O_1O_3^2 - O_2O_3^2$$ ✓

Warning:
Radical center has two cases:

all concurrent
or
all parallel

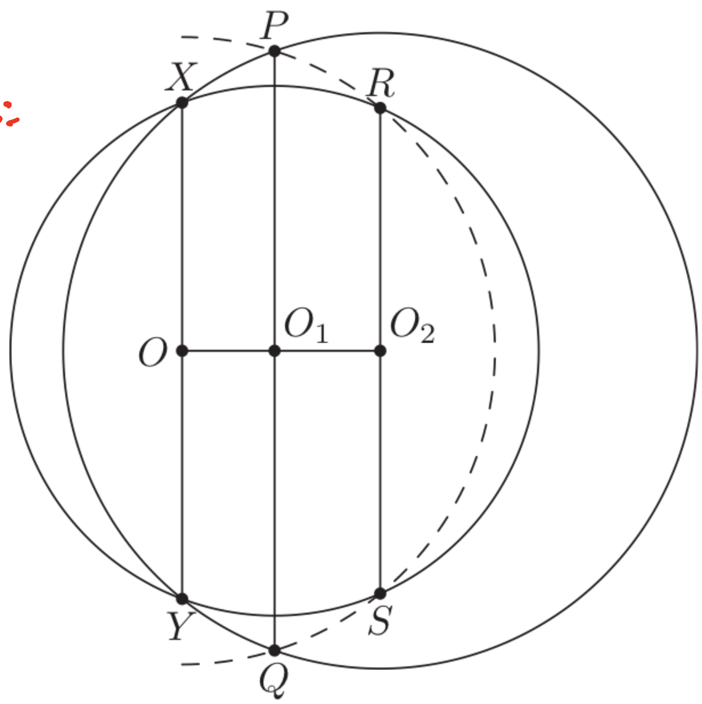
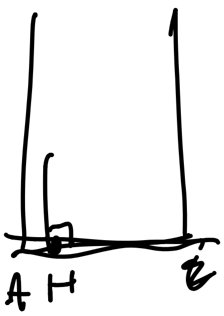
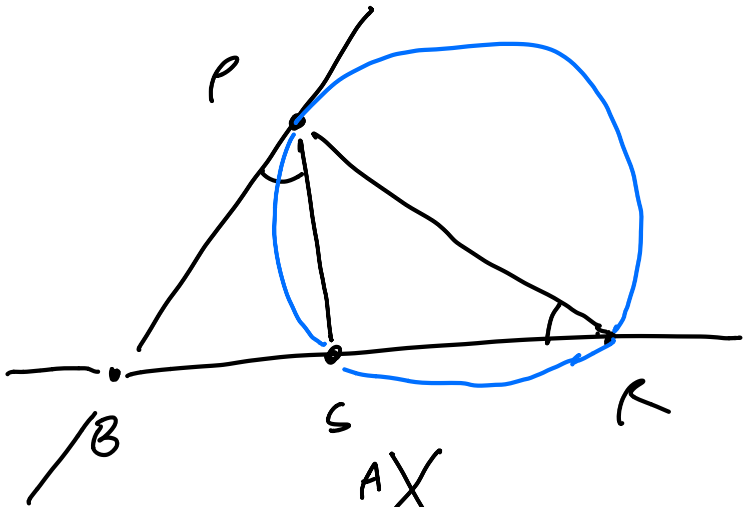


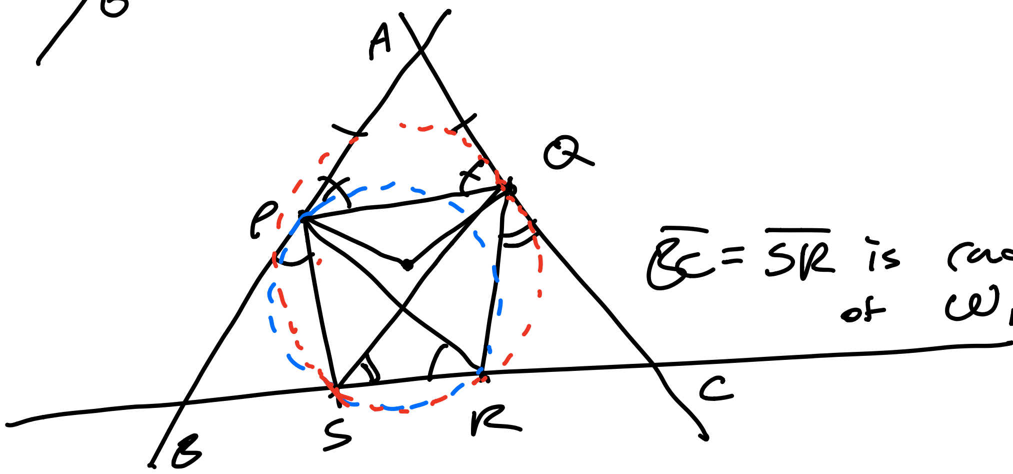
Figure 2.7B. An unnoticed special case.



S

Problem 2.28 (JMO 2012/1). Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic. **Hints**: 435 601 537 122





$\overline{BC} = \overline{SR}$ is radical axis of ω_1, ω_2

$\odot(PRS) = \omega_1 \qquad \odot(QRS) = \omega_2$

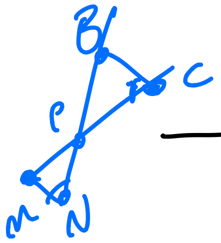
Assume FTSOC $\omega_1 \neq \omega_2$.

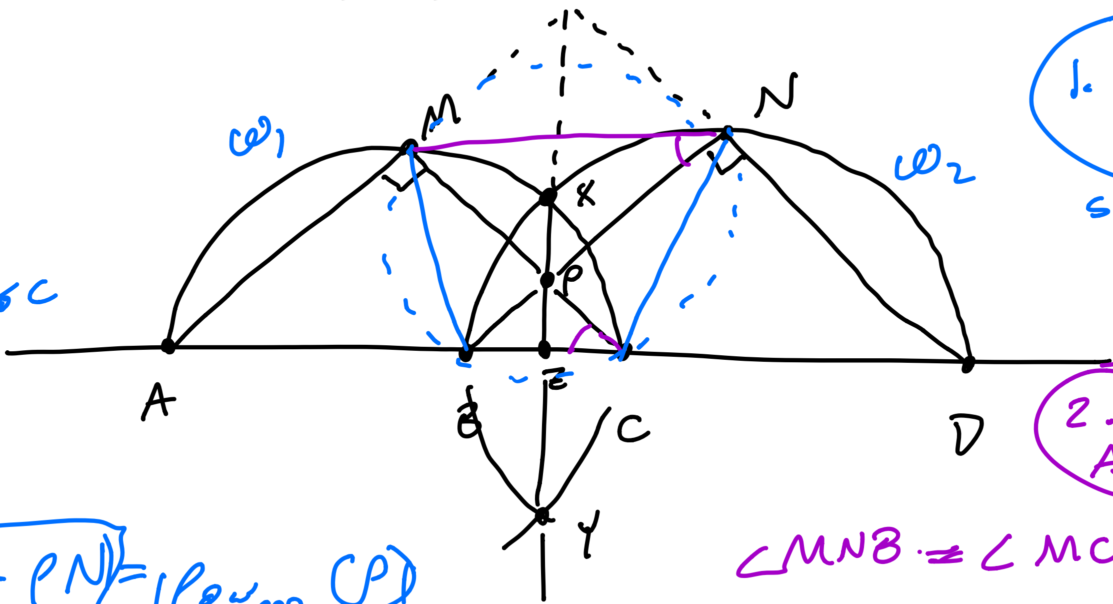
$Pow_{\omega_1}(A) = AP^2 = AQ^2 = Pow_{\omega_2}(A)$

$\rightarrow A \in \overline{BC}$ $\rightarrow\leftarrow$ contradiction.

S

Problem 2.31 (IMO 1995/1). Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent. **Hints**: 49 159 134





1. MNBC cyclic

since PZ radical axis

2. WTS AMND cyclic

since then AM, ND, XY concur @ radical center.

$$\boxed{\begin{aligned}|PB\cdot PN| &= |\mathrm{Pow}_{\omega_2}(P)| \\ &\;\;\| \\ |PC\cdot PM| &= |\mathrm{Pow}_{\omega_1}(P)|\end{aligned}}$$

$\angle MNB = \angle MCB$

$= 90^\circ - \angle MAC$

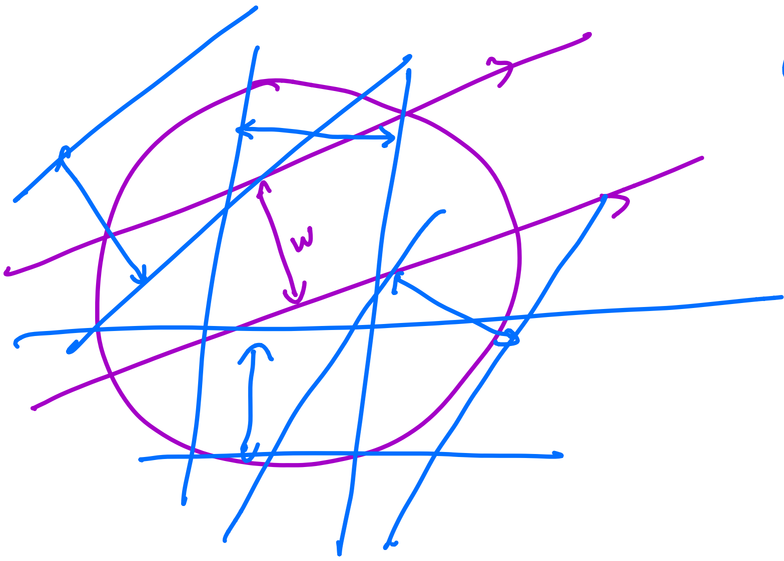
$= \angle MAD$

$\rightarrow \angle MAD + \angle MND$

$= \angle MAD + \angle MNB + 90^\circ$

$= 90^\circ + 90^\circ = 180^\circ$

Measure theory prob.



Prove sum widths of planks $\geq$ diameter of circle.