e and trigonometric functions

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The discovery of the constant e is credited to Jacob Bernoulli in 1683, though its name sake goes to Leonhard Euler.(usually referred to as Euler's number) e is possibly the most used constant in mathematics. Our goal here will be to understand it as much as we can.

Euler's Equation

$$e^{i\theta} = cos(\theta) + isin(\theta) \tag{1}$$

Where $i = \sqrt{-1}$ is the imaginary number.

Problem 1

Assuming x is between 0 and $\pi/2$, graph e^{ix} on the complex plain using equation 1.(vertical axis is imaginary/horizontal is real) Note what x represents on your graph using what you know about the functions sin and cos as they relate to triangles.

hint: where is e^{ix} when $x = 0$? Also $\pi \leftrightarrow 180°$

Such as in eq. 1, we sometimes replace x with θ(the variable which usually represents angles). They may be interchanged throughout the worksheet but θ is often read as an angle while x is usually a real number but can represent anything really. We call $re^{i\theta}$ the polar form of a complex number, and a + bi the cartesian form. $r, \theta, a, b \in \mathbb{R}$

Problem 2

Express the following in both forms of complex numbers.

- $e^{i\pi/3}$
- $\sqrt{2}e^{i11\pi/6}$
- $-\pi$
- $-9+9i$

Sequence Sums and Series

This gives us a notion of what what Euler's equation is doing. It can be thought of as a transformation between the two forms of complex numbers. But is this really the best way of understanding what e is? For a complete understanding you may need more math, in particular calculus and- to extreme extents- abstract algebra and analysis. We can give a definition of e which is consistent with this equation without that much extra knowledge. First we will need one more tool....

Definition 1 Let $\{a_n\}$ define a sequence $a_0, a_1, a_2, a_3, ...$ We call the sum of all the elements in the sequence **the series of** a_n.

$$\sum_{n=0}^{N} a_n = a_0 + a_1 + a_2 + a_3 + ... \tag{2}$$

Where equation 2 is the series. N is the length of the sequence, and if the sequence is infinite in length, then $N \to \infty$.

An infinite series may seem to either be only $\infty, -\infty$, or 0. This is not true in general though. Equation 2 can be thought of as a sequence itself, $A_N = \sum_{n=0}^{N} a_n$ then whatever this sequence converges on we say the sum is equal to. There is rich math concerning sequences and series and when they do and do not converge. We will not concern ourselves too much with that right now, but try to restrict our focus on what is needed for e.

Problem 3

To get a feel for the notation write out the first 3 terms in the following series. Or write the terms as a series.

- $0 + 4 + 8 + 12 + ...$
- $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + ...$
- $\sum_{n=0}^{\infty} \frac{5^n}{n+1}$
- $\sum_{n=0}^{\infty} \frac{\pi}{2}$
- $\sum_{k=0}^{\infty} \frac{a^{2k}+3k}{(5k)!} \qquad (n! = 1 \cdot 2 \cdot ... \cdot n)$

We can define e with just one more theorem. This isn't tough to prove but we would need more knowledge of convergent sequences and series... so we will leave out the proof for now.

Theorem 1

The series below converges for all $z \in \mathbb{C}$. (complex z)

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{3}$$

Definition 2
We define e^z as what the series in Theorem 1 converges to.

Keep in mind that any real number is also complex. So e^x for real x values would be the sum with a real argument.

Problem 4

- Write the sum expression for e
- Show $e^0 = 1$

Problem 5
Express the following as a sum then write the first 3 terms

- $12e^{1+2i}$
- $ie^{\sqrt{2}}$

Recall the exponent rule

$$x^a x^b = x^{a+b} \tag{4}$$

In order for our definition to be consistent with what we usually mean when we write exponents, equation 4 must be satisfied. We'll need a theorem first.

Theorem 2

$$\sum_{i=0}^{\infty} a_i \cdot \sum_{j=0}^{\infty} b_j = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} \tag{5}$$

Problem 6

Find $\sum_{i=0}^{N} a_i \cdot \sum_{j=0}^{N} b_j$ for $N = 0, 1, 2$.

Problem 7

By pairing terms argue Theorem 2 is true as $N \to \infty$.(there's an infinite number of terms being multiplied.) *hint*: try pairing terms using (problem 6) N = 3.

Problem 8

Prove $e^x e^y = e^{x+y}$

hint: binomial theorem

Problem 9

Express $e^{4-\frac{3\pi}{2}i}$ in polar form, then in cartesian form.

Problem 10

Express $\sqrt{10}e^{\sqrt{5}e^{i\pi/15}}$ in polar form, then in cartesian form.

Problem 11

Prove $e^x > 0$ and strictly increasing for $x \in \mathbb{R}$. (Real numbers)

Trigonometric Functions

With problem 10 out of the way we can assert that e^z is probably a good notation for what equation 3 is equal to. With the constant $e = 2.71828...$ being irrational. If this is the case, we should be able to get analytic meaning for *sin* and *cos* using equation 1.

Problem 12

(a) Find a series expression for $sin(\theta)$

(b) Find a series expression for $cos(\theta)$

Problem 13

Prove the following identities. (*hint*: Don't use series definitions.)

(a) $sin(a + b) = cos(a)sin(b) + cos(b)sin(a)$

(b) $cos(a + b) = cos(a)cos(b) - sin(a)sin(b)$

Problem 14

Get series expressions for the following...

- $sin(a)cos(b)$
- $sin(a)sin(b)$
- $cos(a)cos(b)$

Putting these into (a) and (b) from Problem 15 is another way of showing these trigonometric identities.