ORMC: GAME THEORY

OLYMPIAD GROUP 1, WEEKS 5

Problem 1. (Problems with stones). On a table there are n stones. Players A and B play a game where they alternately take stones from the table. At each step, the player to move will take a number s of stones, where s belongs to some fixed set S that A and B have decided on before. Player A always starts the game, and the player who cannot make their move loses. Determine which player wins for each n (assuming perfect play), in the following situations:

(a) $S = \{1\}$ (Yes, this is really easy)
(b) $S = \{1, 2\}$
(c) $S = \{1, 11, 21, 31, \ldots\}$
(d) $S = \{\text{powers of } 2\}$
(e) $S = \{3, 5, 8\}$
(f) $S = \{1, 2, 3\} \cup T$, where T has only odd numbers.
(g) Show that if S is finite, then the pattern eventually repeats.

Problem 2. (Stones but in 2D).

(a) Suppose we are given two piles with m and n stones respectively. Players A and B play a game where they alternatively take any number of stones from exactly one of the two piles (they're not allowed to take stones from both piles on the same turn). Player A starts the game, and the player who cannot make their move loses. For what values of m and n does player B win (assuming perfect play)?

(b) Same problem, but both A and B can take at most 3 stones from a pile of their choice at each turn.

Problem 3.

(a) We have a perfectly circular table. Players A and B take turns alternately, where the player to move places a quarter on the table. The rule is that quarters should never overlap, and should not hang outside the table. If A begins, who has the winning strategy?

(b) Consider the same game, but with three players A, B, C who take turns circularly in this order (A starts). Show that A and C can team up to ensure that B loses.

Problem 4. The number 2020 is written on the board. Players A and B play a game where they take turns alternatively; at each turn, a player can erase one number n from the board and replace it with either of the following:

(i). Two positive integers a, b such that $a + b = n$, or
(ii). Two integers $a, b \geqslant 2$ such that $ab = n$.

Show that the game must eventually end (the player who can't make any more moves loses), and determine which player has a winning strategy (if player A starts).

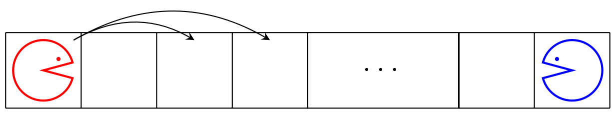
Problem 5.

(a) We are given a 1×2021 board (i.e., 2021 empty slots in a line). At each turn, players A and B (who take turns alternatively) can draw O's in three consecutive slots of the board, provided that all of them are empty before this move. Player A starts, and the player who can't make a move loses. Determine which player has a winning strategy.

(b) Same problem, but with a *circular* 1×2020 board.

Problem 6. A regular n-gon is drawn on the board. Players A, B and C take turns circularly in this order, and player A starts. At each turn, a player must connect two vertices of the n-gon with a line segment, provided that this line segment wasn't already drawn and that it doesn't cross another line segment already drawn (common vertices are okay). As before, the player who can't make a move loses; determine for which values of n player A wins.

Problem 7. Two players A and B play a game on a 1×2020 board. They each have a pacman, situated at one of the endpoints, and facing the other pacman. At each turn, the player moves its pacman 2 or 3 squares in its direction of sight. If one pacman eats the other (i.e. lands on the same equare as the opponent pacman), then that player wins. If the pacmans go past each other without killing, the game is a draw.



If the players take turns alternately, show that no player has a winning strategy.

Problem 8. (BAMO8, 2014, Problem C) Amy and Bob play a game. They alternate turns, with Amy going first. At the start of the game, there are 20 cookies on a red plate and 14 on a blue plate. A legal move consists of eating two cookies taken from one plate, or moving one cookie from the red plate to the blue plate (but never from the blue plate to the red plate). The last player to make a legal move wins; in other words, if it is your turn and you cannot make a legal move, you lose, and the other player has won.

Which player can guarantee that they win no matter what strategy their opponent chooses? Prove that your answer is correct.

Problem *9. (Challenge). All of the positive integers $1, 2, 3, 4, \ldots$ are written on an infinite board. Two players A and B take turns alternatively; at each turn, a player can erase the positive integers from an infinite arithmetic progression, assuming that they're all still written on the board. In other words, they must choose two positive integers a and x such that all the numbers $an + x$ (for $n \in \{0, 1, 2, \ldots\}$) are still on the board, and erase all such numbers. If a player can't make a move, they lose.

(a) Show that the game can only end if there are no numbers left on the board (i.e., otherwise one can always make a move).

(b) Show that if the game ends after the progressions $\{a_1 n + x_1\}_{n \geqslant 0}, \ldots, \{a_k n + x_k\}_{n \geqslant 0}$ have been chosen by the two players, then one must have

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

(c) Show that if a and b are relatively prime positive integers, then two progressions of the form $\{an+x\}_{n\geqslant 0}$ and $\{bn+y\}_{n\geqslant 0}$ cannot be chosen during the same game.

(d) Show that neither player has a winning strategy (that is, both players can guarantee that they don't lose).

Problem *10. (If you know how to play chess) Consider this position. Show that if it's white's turn to move, then black can force a draw, and that if it's black's turn to move, then white can force a win. What changes if we move everything up one square?

Homework I

Problem 1. Like problem 1 from above, but with $S = \{1, 2, \ldots, 2020\}$.

Problem 2. Like problem 2 from above, but the player to move can either take exactly one stone, or two stones from different piles; so

$$(-1, 0) \text{ or } (0, -1) \text{ or } (-1, -1).$$

Problem 3. We have a $2 \times n$ board, and Players A and B take turns in placing dominoes on the board. The rule is that no two dominoes are allowed to overlap, and the dominoes need to line up perfectly with the tiles. Player A moves first, and the player who can't move loses. Who has the winning strategy, depending on n? *Hint: do the $n = 2k$ case first.*

Homework II

Problem 1. Two players A and B take turns in placing coins in the n^2 squares of a $n \times n$ board. At every time, there cannot be two coins in the same square, or in neighboring squares (i.e. sharing a common side). The player who can't move loses. If A starts, who has the winning strategy (depending on n).

Problem 2. Consider an $n \times n$ board, which contains two pacmans on different squares; one belongs to player A, one belongs to player B. They take turns alternately as usual, beginning with A. At each turn, the player must move their pacman one square in either of the 4 directions. If a pacman eats the other (i.e. lands on the same square as the opponent pacman), that player wins.

(a) Say the two pacmans are, initially, in cells that share exactly one common corner. Show that B wins.

(b*) (Challenge) Give a complete description of who (if any) has a winning strategy, in all possible initial arrangements. *Hint: reduce to part (a).* (If you claim for instance that A has a winning strategy for a given position, be sure to show that A can actually win, not only that A can avoid losing)