LAMC Beginners' Circle **November 10, 2013**

Oleg Gleizer
oleg1140@gmail.com

Warm-up

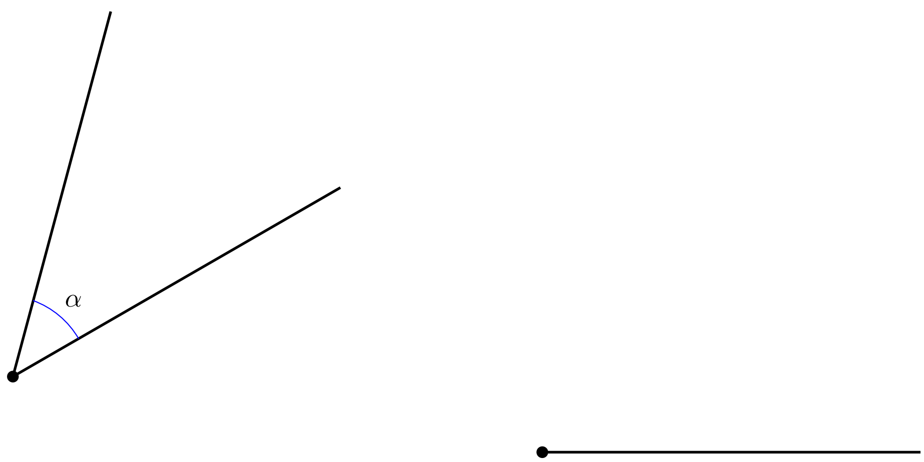
Problem 1 *Can a power of two (a number of the form* 2^n*) have all the decimal digits* $0, 1, \ldots, 9$ *the same number of times?*

Problem 2 *Solve the following cryptarithm.*

$$I^N = DIA$$

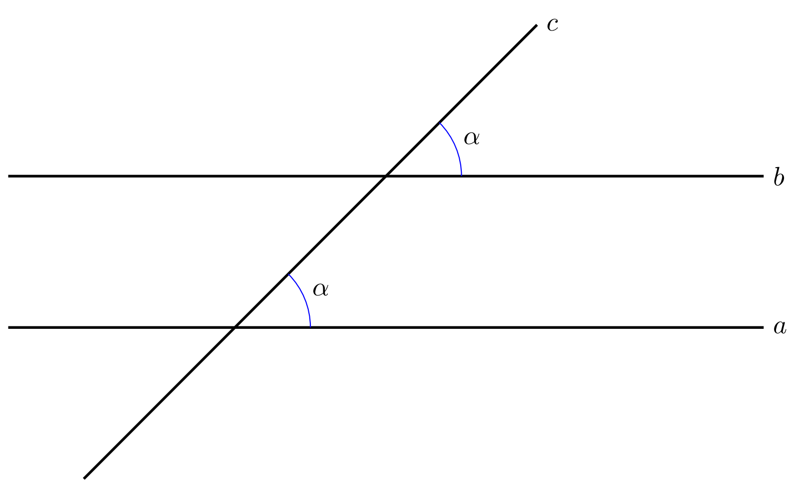
Parallelograms

Problem 3 *Using a compass and a ruler, draw an angle congruent to the given angle α and having the given ray as its side.*



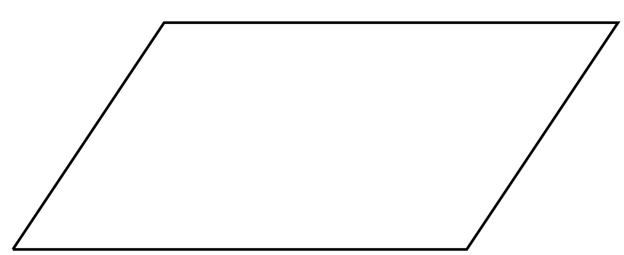
Recall the following.

Proposition 1 *If two distinct straight lines in the Euclidean plane form the angles of equal size with a third straight line in the plane, then they are parallel.*



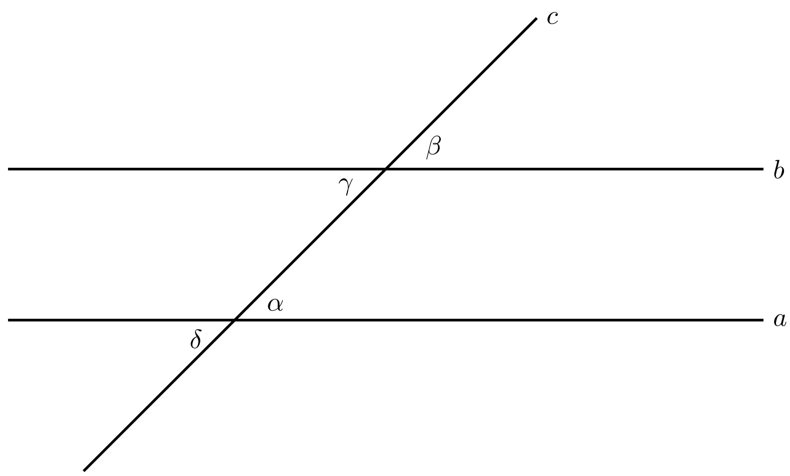
Problem 4 *Using a compass and a ruler, draw a straight line parallel to the given one and passing through the given point.*

Definition 1 *A parallelogram in the Euclidean plane is a quadrilateral that has two pairs of parallel sides.*



parallelogram

Proposition 2 *If the straight lines a and b are parallel, then the angles α, β, γ, and δ they form with the straight line c on the picture below are all congruent to one another.*



Recall the following theorem from the Junior Circle April 15, 2012 handout.[1]

Theorem 1 *Two triangles in the Euclidean plane are congruent if either of the following holds.*

- *Their side lengths are pairwise equal.*

$$|a| = |a'|, \quad |b| = |b'|, \quad |c| = |c'|$$

- *Each of the triangles has an angle congruent to an angle of the other triangle and the lengths of the sides adjacent to the congruent angles are pairwise equal.*

$$\alpha \cong \alpha', \quad |b| = |b'|, \quad |c| = |c'|$$

[1] http://www.math.ucla.edu/~radko/circles/lib/data/Handout-345-431.pdf

• *The triangles have one side of equal length each, and the adjacent angles are pairwise congruent.*

$$|c| = |c'|, \quad \alpha \cong \alpha', \quad \beta \cong \beta'$$

To prove the following theorem, we will use Proposition 2 and Theorem 1 as tools. We are also going to use the Claim - Reason charts studied in the April 28, 2013 Junior Circle handout.[2]

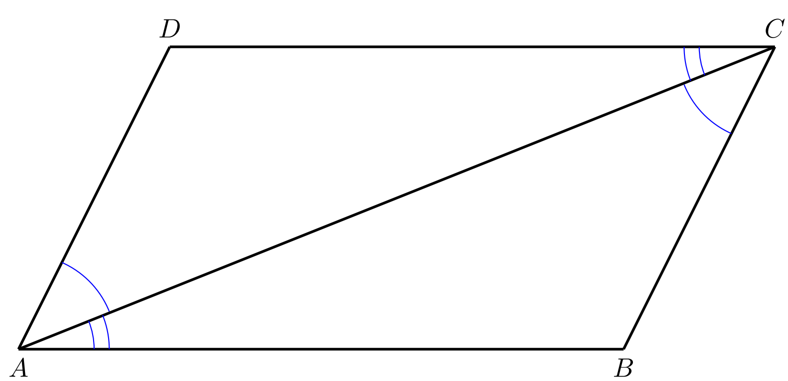
Theorem 2 • *Opposite sides of a parallelogram have equal length.*

- *Opposite angles of a parallelogram are congruent.*

- *Diagonals of a parallelogram split each other in halves.*

The proof of the first statement of Theorem 2 is given below. You will be asked prove the remaining two parts in the subsequent problems.

Consider the following picture.



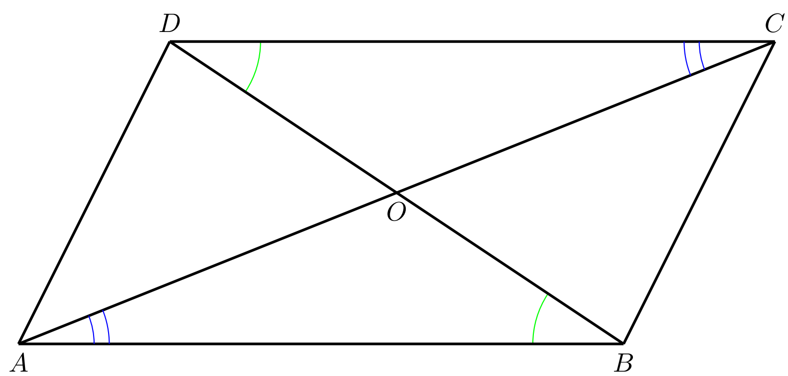
[2]http://www.math.ucla.edu/~radko/circles/lib/data/Handout-517-630.pdf

Claim	**Reason**
AC is the common side of the triangles ABC and ACD.	Given.
$(AD) \parallel (BC)$	Definition of parallelogram.
$\angle CAD \cong \angle ACB$	Proposition 2.
$(AB) \parallel (CD)$	Definition of parallelogram.
$\angle ACD \cong \angle BAC$	Proposition 2.
$\triangle ABC \cong \triangle ACD$	Third part of Theorem 1: $\lvert AC\rvert = \lvert AC\rvert$, $\angle CAD \cong \angle ACB$, $\angle ACD \cong \angle BAC$
$\lvert AB\rvert = \lvert CD\rvert$, $\lvert AD\rvert = \lvert BC\rvert$	$\triangle ABC \cong \triangle ACD$

Q.E.D.

Problem 5 *Use the Claim-Reason chart to prove the second statement of Theorem 2.*

Problem 6 *To prove the last statement of Theorem 2, consider the triangles DOC and AOB on the following picture.*

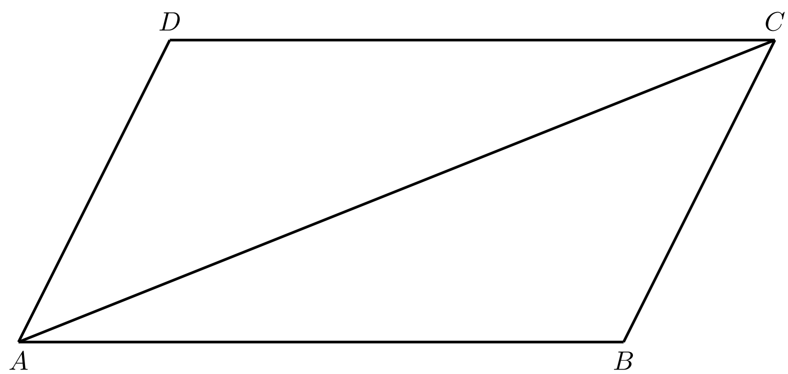


Then use the Claim-Reason chart.

Proposition 3 *A quad having pairs of opposite sides of equal length is a parallelogram.*

The proof of this important proposition is split into the following two problems.

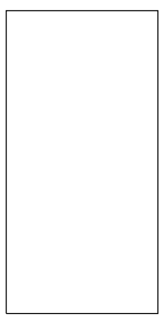
Problem 7 *For the quad $ABCD$ below, it is given that $|AD| = |BC|$ and $|AB| = |CD|$. Use the Claim - Reason chart to prove that $\angle CAD \cong \angle ACB$ and $\angle ACD \cong \angle BAC$.*

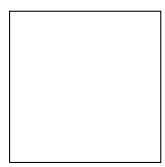


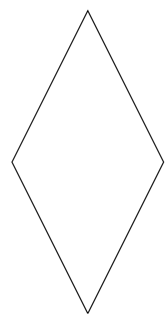
Problem 8 *Use Problem 7 and Proposition 1 to finish the proof of Proposition 3. Fill out the corresponding Claim - Reason chart.*

Definition 2

- *A rectangle is a quadrilateral with all four angles congruent to one another.*
- *A square is a rectangle with all four sides of equal length.*
- *A rhombus is a quad with all four sides of equal length.*


a rectangle


a square


a rhombus

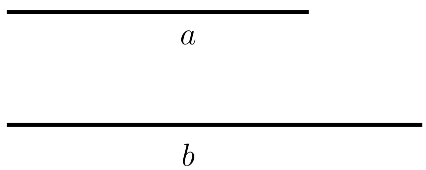
A rhombus is sometimes called a *diamond*. A rectangle is occasionally called an *oblong*, the word generally meaning something that is longer than it is wide. Note that a square is a rectangle and a rhombus at the same time.

Problem 9 *Prove that a rectangle is a parallelogram. Use the Claim-Reason chart.*

Problem 10 *Prove that a rhombus is a parallelogram. Use the Claim-Reason chart.*

Problem 11 *Prove that diagonals of a rhombus intersect at the right angle. Use the Claim - Reason chart.*

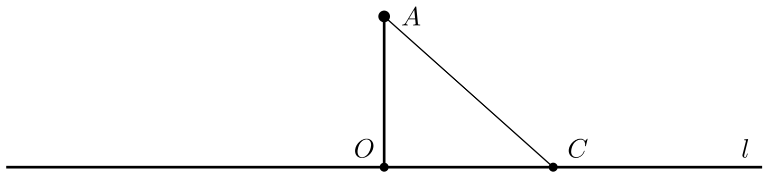
Problem 12 *Use a compass and a ruler to draw the right triangle with the following legs in the space below.*



Definition 3 *Two straight lines intersecting at the right angle are called orthogonal, or perpendicular, to each other.*

For example, the diagonals of a rhombus are always orthogonal.

Theorem 3 *In the Euclidean plane, the shortest path from a point to a straight line is the perpendicular from the point to the line.*

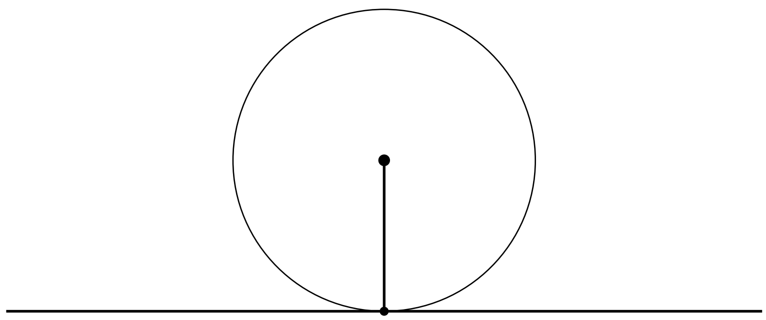


Problem 13 *Let* $(AO) \perp l$ *on the picture above. Let* C *be any other point on* l*. With the help of the Pythagoras' Theorem, prove that* $|AC| > |AO|$*. Use the Claim - Reason chart.*

Theorem 4 *For any point and straight line in the Euclidean plane, there exists a unique straight line passing through the point orthogonal to the original line.*

Problem 14 *Prove Theorem 4.*

Definition 4 *A straight line is called tangent to a circumference, if they intersect at one point.*



Theorem 5 *A line tangent to a circumference is orthogonal to the radius drawn from their common point to the circumference center.*

Problem 15 *Prove Theorem 5.*

Tired of geometry?

Problem 16 *Prove the following polynomial identity.*

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \quad (1)$$

Problem 17 *Prove that in any base* $b > 3$*, the number* 1331_b *is a perfect cube.*

Problem 18 *Without using a calculator, find the cubic root of the number* $16c8_{16}$. *Hint: formula 1 may help.*

Definition 5 *Two integers are called co-prime, if their greatest common divisor equals one.*

For example, a prime number is co-prime with any other integer that is not a power of the prime.

Problem 19 *Without using a calculator, decide whether the decimal numbers* 11 *and* $9,182,371$ *are co-prime. Explain your decision.*

Problem 20 *One chooses* $n+1$ *numbers between* 1 *and* $2n$. *Show that she/he has selected two co-prime numbers.*

Homework

Please refresh your knowledge of the geometry of masses (Junior Circle April 7, 2013 handout, starting from page 5 [3]) if you have studied it before. If not, try to learn as much as you can. Then solve the following.

Problem 21 *One-pound weights are placed in the vertices of a parallelogram. Find its center of mass.*

[3] http://www.math.ucla.edu/~radko/circles/lib/data/Handout-510-612.pdf