# Egyptian Fractions: Part I

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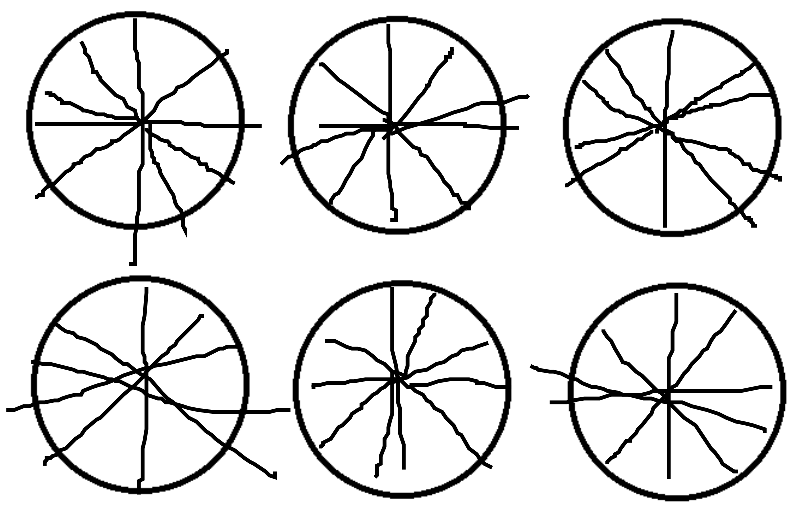
## 1 Cutting Cakes

1. Imagine you are a teacher. Your class of 10 students is on a field trip to the bakery. At the end of the tour, the baker gives you 6 round cakes to be divided evenly among the 10 students.

   (a) Let each circle below represent one cake. Show how you would cut the cakes into pieces. Represent one cut with a straight line through the center of the cake.



**Cut each cake into tenths. Alternatively, cut each cake into fifths.**

(b) How many pieces would each student receive?

**6 pieces (3 pieces if cut into fifths)**

(c) What amount of cake would each student receive?

**6/10 or 3/5 of a cake**

(d) Suppose it would take you 1 minute to hand out each piece of cake. How long would it take to hand out cake to all students?

**60 minutes if cut into tenths, 30 minutes if cut into fifths**

If you're like me, you split each cake into 10 pieces, each piece having size $\frac{1}{10}$ of a cake. Then, you gave each student 6 of these pieces. This is one solution, but it is not perfect. This solution involves lots of small pieces for each student. Lots of small pieces are time-consuming to distribute and also produce lots of crumbs. There is another way to solve the problem with fewer, larger pieces for each student (and less crumbs)!

2. Again, suppose we have 6 cakes to divide evenly among 10 students. We will try to give each student fewer, and thus larger, pieces. In order to ensure all pieces are even, the only cuts we make are straight lines through the center of the cake.

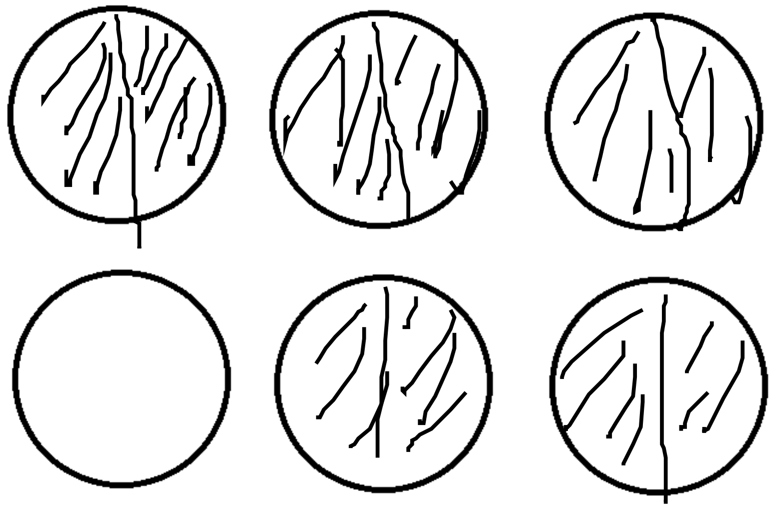
    (a) The largest piece we could give to a student is one whole cake. This is accomplished with zero cuts. Is there enough cake to do this?

    **No. This would require 10 cakes.**

    (b) The second largest piece we could give to a student is $\frac{1}{2}$ of a cake. Is there enough cake to do this?

    **Yes, this requires only 5 cakes.**

(c) On the circles below, draw straight lines to represent how you would cut the cakes in order to give each student $\frac{1}{2}$ of a cake. Shade in all of the pieces you hand out.



(d) How many cakes would be left over?

**One cake**

(e) How can you split the leftovers evenly among the 10 students? Draw a picture or write down the appropriate fraction of cake that each student receives.

**Give each student 1/10 of a cake**

(f) What is the total amount of cake that each student receives? Give your answer as the sum of two fractions.

**1/2 + 1/10**

(d) Again, suppose it takes you one minute to hand out each piece of cake. How long would it take to hand out cake to all students?

**20 minutes**

## 2 Unit Fractions and Ancient Egyptians

When an object is divided into equal pieces, we call each of these equal pieces a **unit fraction** of the original object. The unit fractions are the fractions that can be written as $\frac{1}{n}$ for some integer $n$. $\frac{1}{2}$, $\frac{1}{10}$, and $\frac{1}{67}$ are all examples of unit fractions. To better understand unit fractions, we will examine what kinds of fractions we can make out of them.

3. Compute the following sums of unit fractions.

   (a) $\frac{1}{2} + \frac{1}{10} =$ **6/10 or 3/5**

   (b) $\frac{1}{5} + \frac{1}{10} =$ **3/10**

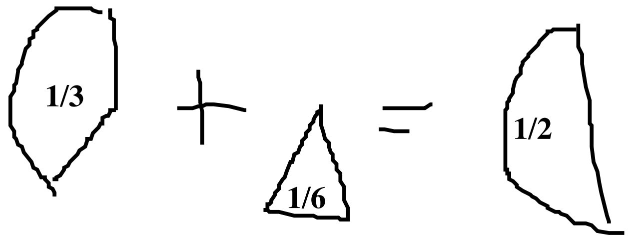
   (c) $\frac{1}{3} + \frac{1}{5} + \frac{1}{6} =$ **21/30 or 7/10**

(d) $\frac{1}{2} + \frac{1}{15} + \frac{1}{100} =$ **173/300**

(e) $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} =$ **129/120 or 43/40**

4. (a) Examine the unit fraction pieces in front of you. Using the pieces, find two **different** unit fractions whose sum is a unit fraction.

- Draw a picture to show your solution.




- Add up the two fractions to show that your solution works.

**1/3 + 1/6 = 3/6 = 1/2**

(b) Find another example of two different unit fractions whose sum is a unit fraction.

**1/4 + 1/12 = 1/3**

(c) Find at least two pairs of unit fractions whose **difference** is a unit fraction.

**1/2 - 1/3 = 1/6**

**1/3 - 1/4 = 1/12**

**1/4 - 1/5 = 1/20, etc.**

As you can see, many different fractions can be represented by sums of distinct unit fractions. Representing fractions in this way has benefits, as we saw in our cake-cutting problem. It provides bigger pieces which are easier to distribute (and don't make so many crumbs).

The Ancient Egyptians, famous for their mathematics, wrote all of their fractions as sums of distinct unit fractions. Their notation for unit fractions went like this:

$$\dot{2} = \frac{1}{2}, \quad \dot{3} = \frac{1}{3}, \quad 1\dot{0}0 = \frac{1}{100}$$

and so on. For the remainder of this packet, we will use the modern notation for unit fractions as to avoid confusion.

For fractions like $\frac{3}{4}$, an Egyptian would use a sum of unit fractions, and write $\frac{1}{2} + \frac{1}{4}$. Notice, one would not write $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$. The Ancient Egyptians were very smart, and knew that repeating the same fraction more than once would only result in a *longer* representation, and never a shorter one. We will prove this result in Part II of this packet.

A representation of a fraction as a sum of distinct unit fractions is said to be an **Egyptian Fraction Representation** (EFR) for the given fraction. Note that it is not necessarily *the* EFR for a given fraction. We have not yet determined whether each fraction has a single unique EFR. We will work to answer the following **BIG QUESTIONS**:

**1)** Does every fraction have at least one EFR? If so, we must prove this. If not, we must find an example of a fraction which has **no** EFR.

**2)** How do we find an EFR for a given fraction?

**3)** Are there several EFRs for a single given fraction?

Big Question **1)** turns out to be more difficult than **2)** or **3)**, so we will save it for the end of our exploration. For now, we will discover strategies for finding one, or possibly multiple, EFRs of a given fraction. Later we will tackle the question of whether **every** fraction can be represented in this form.

# 3 Greedy Greg finds EFRs

To learn more about question **1)**, we will go back in time to visit an Ancient Egyptian boy named **Greedy Greg**.

Greedy Greg is learning about fractions in school. He knows his unit fractions but has not learned to add them together. On some nights, Greg's mother brings him a fraction of a cake for dessert. She wants Greg to learn to add fractions together, so she offers him a reward.

After dessert, if Greg can tell his mother what fraction of cake he ate (in Egyptian Fraction Representation, of course) then she will bring home more cake the following night. For example, if Greg's mother brought home $\frac{6}{10}$ of a cake, after he eats, Greg could earn a reward by telling his mother that he has eaten $\frac{1}{2} + \frac{1}{10}$ of a cake.

Greedy Greg's plan is to eat the cake in **unit fraction** pieces, always taking the **biggest** piece he can and being sure to **never repeat** fractions. Then, after dessert, he can simply tell his mother the sizes of the individual pieces that he ate.

For example, on Sunday night his mother brings home $\frac{6}{10}$ of a cake. Since $\frac{1}{2}$ is the **largest** unit fraction less than or equal to $\frac{6}{10}$, Greg asks her to serve him $\frac{1}{2}$ of a cake. After eating this first piece, he sees that there is $\frac{1}{10}$ of a cake remaining. Since $\frac{1}{10}$ is the **largest** unit fraction less than or equal to $\frac{1}{10}$, Greg asks his mother to serve him the remaining $\frac{1}{10}$ of a cake. He then correctly tells his mother that he has eaten $\frac{1}{2} + \frac{1}{10}$ of a cake, earning his reward.

In Part II, we will prove that Greg's strategy works no matter what fraction of a cake his mother brings home. This will then prove that every fraction has at least one EFR. Until we prove this, our results will only apply to fractions which have at least one EFR. We will begin by practicing Greedy Greg's method for finding EFRs.

5. It is Monday night and this time Greedy Greg's mother brings home $\frac{4}{5}$ of a cake.

   (a) What amount of cake will Greg ask for first? Give your answer as a fraction.

   **1/2 of a cake**

   (b) How much cake will there be left?

   **3/10 of a cake**

(c) Greg has eaten his first piece of cake. What amount of cake will Greg ask for next?

**1/4 of a cake**

(d) How much cake will there be left?

**1/20 of a cake**

(e) Greg has eaten his second piece of cake. What amount of cake will Greg ask for next?

**1/20 of a cake**

(f) Will there be any cake left?

**No**

(g) Based on the information above, write down one EFR for $\frac{4}{5}$.

**4/5 = 1/2 + 1/4 + 1/20**

6. It is now Tuesday night and Greedy Greg's mother brings home $\frac{29}{30}$ of a cake for dessert.

   (a) Pretend you are Greedy Greg and come up with a plan to finish your dessert. Indicate what amount of cake you will ask for at each step.

      **Step 1: Ask for 1/2 of a cake. 14/30 or 7/15 of a cake remains.**

      **Step 2: Ask for 1/3 of a cake. 2/15 of a cake remains.**

      **Step 3: Ask for 1/8 of a cake. 1/120 of a cake remains.**

      **Step 4: Ask for 1/120 of a cake. No cake remains.**

   (b) Does $\frac{29}{30}$ have an EFR? If so, what is it?

      **Yes. 29/30 = 1/2 + 1/3 + 1/8 + 1/120**

7. For an Ancient Egyptian like Greedy Greg, it is easy to determine the largest unit fraction less than or equal to a given fraction. Those of us who have not spent our entire lives looking at unit fractions need an algorithm to find this largest possible unit fraction. For exapmle, it's not immediately clear what the largest unit fraction less than or equal to $\frac{47}{901}$ is. If we are given a fraction $\frac{a}{b}$, we would like to find the **smallest** integer $n$ (and thus the largest unit fraction $\frac{1}{n}$) such that

$$\frac{1}{n} \leq \frac{a}{b}$$

This statement is equivalent to the following double inequality,

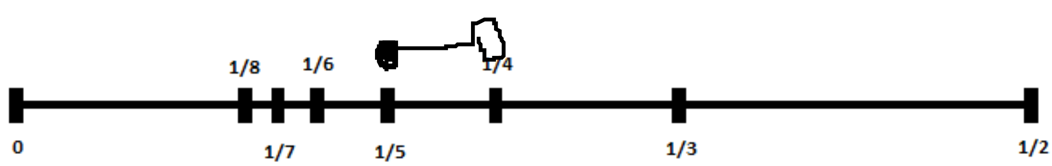
$$\frac{1}{n} \leq \frac{a}{b} < \frac{1}{n-1}$$

which says that $\frac{a}{b}$ is greater than or equal to $\frac{1}{n}$ and also less than $\frac{1}{n-1}$.

(a) How do we know that $\frac{a}{b}$ is less than $\frac{1}{n-1}$?

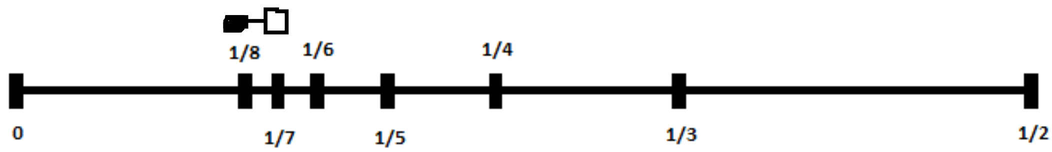
**It must be because if it were not, then 1/(n-1) would be a unit fraction larger than 1/n and less than or equal to a/b. This contradicts the fact that 1/n is the LARGEST unit fraction less than or equal to a/b.**

(b) Suppose $\frac{1}{5} \leq \frac{a}{b} < \frac{1}{4}$. On the number line below, indicate where the fraction $\frac{a}{b}$ may be located. Below the number line, write the largest unit fraction less than or equal to $\frac{a}{b}$.




**1/5**

(c) Suppose $\frac{1}{8} \leq \frac{a}{b} < \frac{1}{7}$. On the number line below, indicate where the fraction $\frac{a}{b}$ may be located. Below the number line, write the largest unit fraction less than or equal to $\frac{a}{b}$.




**1/8**

(d) Suppose $\frac{a}{b}$ lies in the following interval:


What is the largest unit fraction less than or equal to $\frac{a}{b}$?

**1/n**