LAMC High School Fall 2011

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Intro to geometry, or is your world flat?

Part 3

Elliptic Geometry[1]

"Glue" antipodal points P and P' of a sphere into one point. Result: a real projective plane $\mathbb{RP}^2$. Its geometry is locally spherical, but globally closer to the geometry of the euclidean plane $\mathbb{R}^2$ than spherical geometry due to the absence of antipodes in the former. For example, any two straight lines on $\mathbb{RP}^2$ intersect at one point. What's more, all the axioms of the Euclidean geometry are true for the elliptic one, except for the fifth postulate. There are no parallel lines in elliptic geometry!

Projective spaces are very important. For example, a quantum particle is a point of an infinite-dimensional projective space over complex numbers. A point of a projective space $\mathbb{P}^n$ is a straight line of the ambient linear space $\mathbb{V}^{n+1}$. For example, points of $\mathbb{RP}^2$ are straight lines in $\mathbb{R}^3$. Take a sphere centered at zero of the linear space and each line will mark two antipodal points on it, yielding Riemann's construction.

Topologically, $\mathbb{RP}^2$ is a sphere with a hole and a Möbius strip glued to the boundary of the hole. In particular, $\mathbb{RP}^2$ is a one-sided surface.

[1]Invented by Riemann, 1826 – 1866.

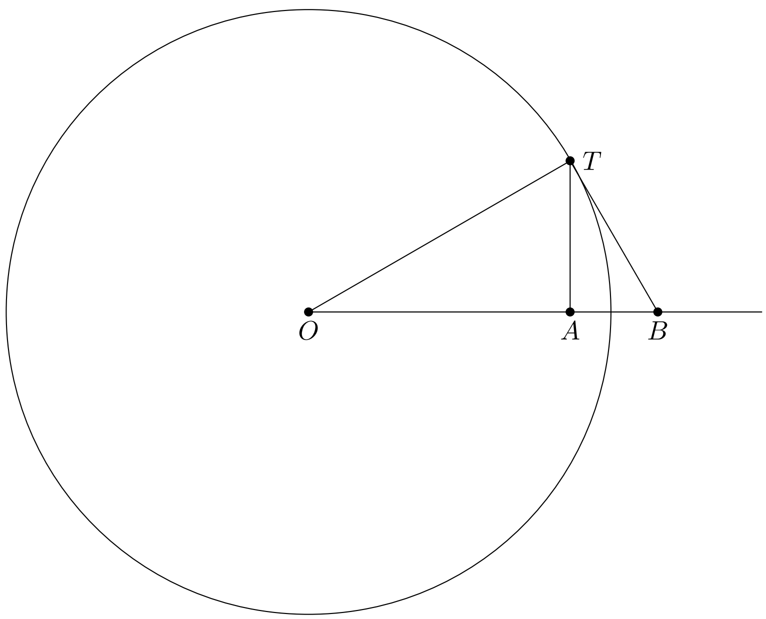
Toolbox

Let $\bar{\mathbb{R}}^2$ be $\mathbb{R}^2$ with an extra point at ∞.

Definition 1 *An inversion $\mathcal{R}_{O,r}$, a.k.a. a reflection in the circle centered at O of radius r, is a transformation of $\bar{\mathbb{R}}^2$ interchanging the points*

1. *A and B of every ray emanating from O such that $|OA||OB| = r^2$;*

2. *O and ∞.*



Since (TB) is tangent to the circumference, $\angle OTB = 90°$. Thus, the triangles OAT and OBT are similar to each other, rendering the following formula.

$$\frac{|OA|}{r} = \frac{r}{|OB|}$$

Properties of inversions.

1. Inversions are bijections of $\bar{\mathbb{R}}^2$.
2. Inversions are involutions, $\mathcal{R}^2 = Id$.
3. Inversions map straight lines and circles to straight lines and circles.

Problem 1 *An inversion maps a straight line passing through O to itself ("inside out" – O changes places with ∞).*

Problem 2 *Let $\mathcal{R}(A) = A'$, $\mathcal{R}(B) = B'$. Prove that $\triangle OAB$ and $\triangle OA'B'$ are similar to each other.*

Problem 3 *Prove that an inversion maps a straight line not passing through O to a circumference passing through O.*

Problem 4 *Prove that an inversion maps a circumference passing through O to a straight line not passing through O.*

Problem 5 *Prove that an inversion maps a circumference not passing through O to a circumference not passing through O.*

Definition 2 *An angle between two circumferences (a circumference and a straight line) is the angle between the straight lines tangent to their common point.*

Problem 6 *Prove that inversion preserves the angles between circumferences and straight lines.*

It follows from the above that inversions preserve angles between any smooth lines. Maps having this property are called *conformal*.

Problem 7 *Prove that an inversion maps a circumference or straight line orthogonal to the inversion circumference to itself.*

Problem 8 *Let A be a point inside a circumference centered at O. Prove that there exists a circumference orthogonal to the original one such that an inversion in the latter maps A to O.*

Problem 9 *Let A and B be some inside points of a circumference centered at O not lying on the same diameter. Prove that there exists a unique circumference passing through the points and orthogonal to the original circumference.*

Problem 10 *Let A be an inside point of a circumference centered at O. Let B be a point of the circumference not lying on the same diameter as A. Prove that there exists a unique circumference passing through the points and orthogonal to the original circumference.*

Problem 11 *Let A and B be points of a circumference centered at O. Prove that there exists a unique circumference passing through A and B and orthogonal to the original circumference.*

Problem 12 *Let A be a point inside the circumference $\mathcal{C}_1$. Let $\mathcal{C}_2$ be a circumference orthogonal to $\mathcal{C}_1$. Prove that there exists a unique circumference passing through A and orthogonal to $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Hyperbolic Geometry

$$\text{Let } \mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

We shall call $\mathbb{H}^2$ the *hyperbolic plane* (a.k.a. the Lobachevsky plane, after the Russian mathematician Lobachevsky, 1792 – 1856, who was the first to discover it). The boundary circumference of $\mathbb{H}^2$ is called the *absolute*. Note that the points of the absolute do not belong to $\mathbb{H}^2$.

Points: Euclidean points of the open unit circle in $\mathbb{R}^2$.

Straight lines: arcs of circumferences and segments of straight lines orthogonal to the absolute.

Theorem 1 *For any two distinct points of $\mathbb{H}^2$, there exists a unique straight line passing through them.*

Proof — See Problem 9.

Definition 3 *Two lines in $\mathbb{H}^2$ are orthogonal, if they are orthogonal in the underlying Euclidean geometry of $\mathbb{R}^2$.*

Theorem 2 *For a straight line and a point in $\mathbb{H}^2$, there exists a unique straight line passing through the point and orthogonal to the original line.*

Proof — See Problem 12.

In fact, all the axioms of the Euclidean geometry are true for the hyperbolic one, except for the fifth postulate.

Theorem 3 *For any straight line in $\mathbb{H}^2$ and any point away from it, there exist infinitely many straight lines passing through the point and not intersecting the original line.*

Problem 13 *Prove the theorem.*

Theorem 4 *For any triangle in $\mathbb{H}^2$ having the area S and angels α, β, and γ measured in radians,*

$$S = \pi - (\alpha + \beta + \gamma).$$

In his famous book *Science et Hypothése*, Henri Poincaré describes the physics of a flat hyperbolic world and the physical theories that its inhabitants would create. The center of $\mathbb{H}^2$ has the temperature of $100°F$. The temperature decreases linearly to absolute zero at the absolute. The lengths of objects (including the tiny bugs inhabiting the world) are proportional to the temperature.

Exploring their world the bugs will discover that

1. it's infinite – they can never reach the boundary.

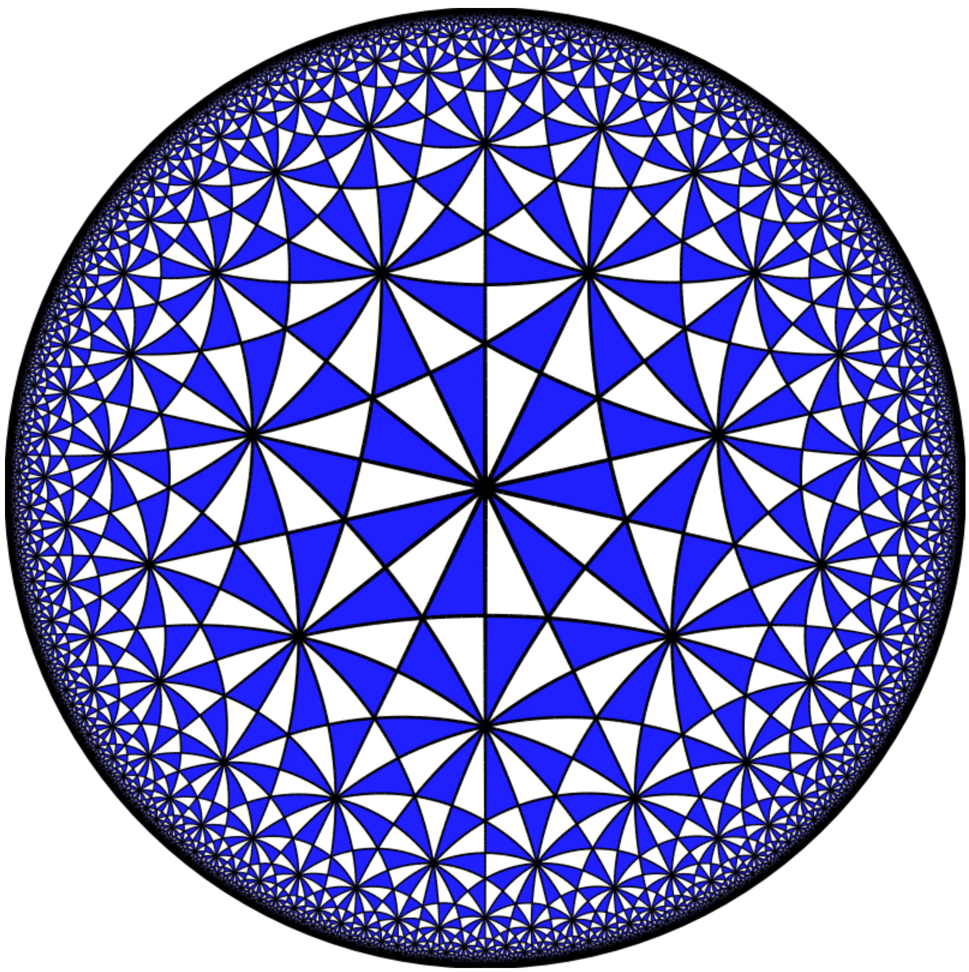
2. It has constant temperature. Indeed, thermometers are based on different expansion coefficients of various materials, but since the lengths of all objects change similarly with temperature, the thermometers will gives the same measurement all over the world.

3. Straight lines, i.e. shortest paths between the points. They will discover all the postulates of the Euclidean geometry to hold for what they perceive as straight lines except for the fifth one. The fifth postulate replaced with the statement of Theorem 3, they will further discover the above formula for the sum of the angles of a triangle, etc.

Eventually, the bugs will come to the conclusion that they live in an infinite flat universe with constant temperature governed by the laws of hyperbolic geometry. But this not true – their universe is a finite disk, its temperature is variable and the underlying geometry is Euclidean, not hyperbolic!

The physical model described above shows not only that the truth about the universe cannot be discovered, but that it makes no sense to speak of any "truth" or approximation of "truth" in science. The bugs are perfectly right to use hyperbolic geometry as the foundation of their physics.

The pattern below shows the tiling of the Lobachevsky plane with triangles. To the inhabitants of the plane, the triangles seem to be of the same size and have all the sides straight as well!



The following drawing, called “Angels and devils”, by the famous Dutch graphic artist Maurits Cornelis Escher, 1898 – 1972, was obviously inspired by the above tiling.