LAMC Advanced Circle **April 23, 2017**

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Problem 1 *Bring the equation*

$$z^3 - 3z^2 + 6z - 2 = 0. \tag{1}$$

to the depressed form.

Problem 2 *Find a root of the depressed form of (1).*

Problem 3 *Find the remaining two roots of the depressed form of (1).*

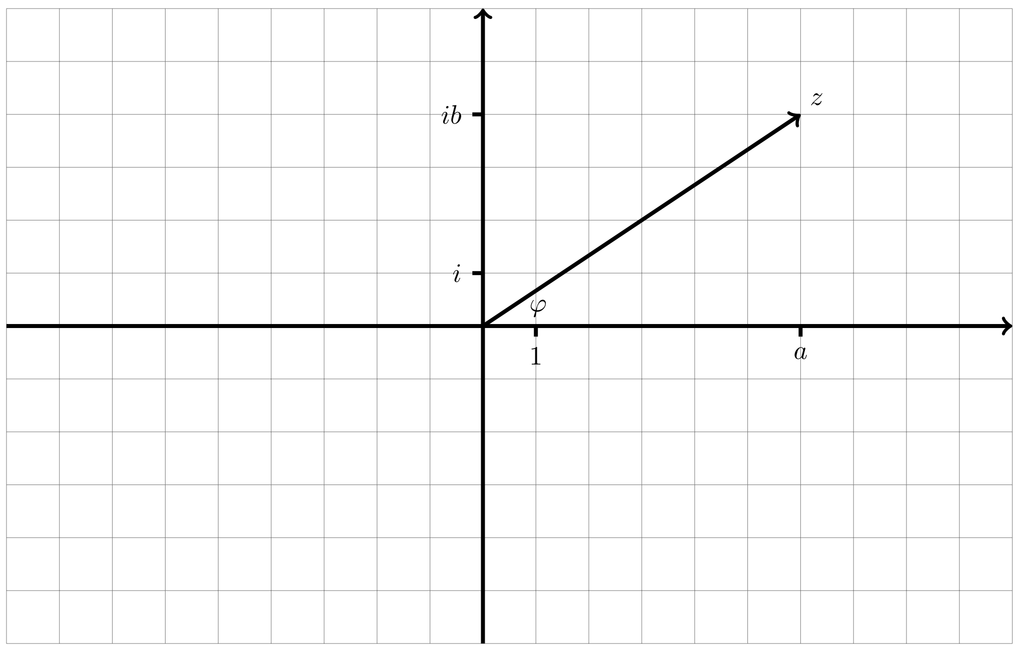
Question 1 *Do you feel that for solving a generic cubic equation, we need another tool in addition to the Cardano formula?*

Complex numbers in the polar form

Below is a picture of a complex number

$$z = a + ib$$

represented as a vector in the complex plane, $\mathbb{C}$.



Let φ be the angle, measured in radians, between z and the real axis, positively oriented the standard way. Recall that

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}. \tag{2}$$

Therefore,

$$a = |z| \cos\varphi \quad \text{and} \quad b = |z| \sin\varphi.$$

In other words,

$$z = |z|(\cos\varphi + i \sin\varphi). \tag{3}$$

We will not prove the following formula, known as *Euler's identity*, in this handout.

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \tag{4}$$

Note that a particular case, $e^{i\pi} = -1$, rewritten as

$$e^{i\pi} + 1 = 0, \tag{5}$$

binds together five most fundamental mathematical constants, 0, 1, i, e, and π.

Combining (3) and (5) produces

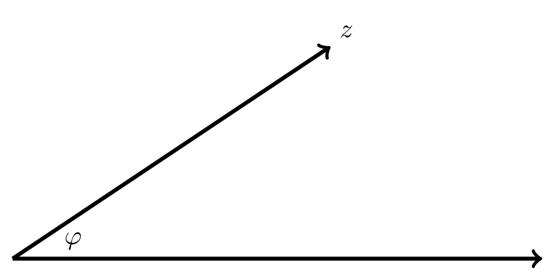
$$z = |z|e^{i\varphi}.$$

To better see the new, *polar*, coordinates, let us set $r = |z|$. Then

$$z = a + ib = re^{i\varphi}.$$

The first representation, $z = a + ib$, is in rectangular coordinates. The second, $z = re^{i\varphi}$, is in the polar coordinates (r, φ).

Instead of a pair of coordinate axes needed for setting up rectangular coordinates, polar coordinates require only a ray.



To construct the vector z, one takes a second ray forming the angle φ with the original one. The terminal point of the vector lies on the second ray at the distance $r = |z|$ from the origin. The following formulas are a go-between the rectangular and polar coordinates.

$$\begin{aligned} a &= r\cos\varphi & \qquad r &= \sqrt{a^2+b^2} \\ b &= r\sin\varphi & \qquad \varphi &= arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right) \end{aligned} \tag{6}$$

Note that in the US $arccos(x)$ is usually denoted as $\cos^{-1}(x)$. The latter notation is ambiguous, because $\cos^{-1}(x)$ can also mean $1/\cos x$.

Problem 4 *Represent the complex number* $\sqrt{3}e^{i\pi/6}$ *in the* $a+ib$ *form.*

Problem 5 *Represent the complex number* $\sqrt{3}e^{i5\pi/6}$ *in the* $a+ib$ *form.*

Problem 6 *Represent the complex number* $\sqrt{3} + i$ *in the polar form.*

Problem 7 *Represent the complex number* $\sqrt{3} - i$ *in the polar form.*

Problem 8 *Prove that* $e^{i2\pi} = 1$.

Problem 9 *Prove that if* $z = re^{i\varphi}$, *then* $\bar{z} = re^{-i\varphi}$.

Problem 10 *For the complex numbers* $p = 2e^{i2\pi/3}$ *and* $q = 5e^{-i\pi/3}$, *find the following.*

- $pq =$

- $p \div q =$

- $p^2 =$

- $p^3 =$

- $p^4 =$

- $p^8 \div q^{10} =$

Polar coordinates are more convenient for multiplication, division, and powering of complex numbers than rectangular coordinates. If $z_1 = r_1 e^{i\varphi_1}$, $z_2 = r_2 e^{i\varphi_2}$ and $z = re^{i\varphi}$, then $z_1 z_2 = r_1 r_2 e^{i(\varphi_1+\varphi_2)}$, $z_1 \div z_2 = r_1 r_2 e^{i(\varphi_1-\varphi_2)}$, and $z^n = r^n e^{in\varphi}$.

Problem 11 *Consider the product $z_1 z_2$ of the complex numbers $z_1 = e^{i\alpha}$ and $z_2 = e^{i\beta}$ to prove the following very important trigonometric identities.*

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Problem 12 *Prove that*

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

Problem 13 *Prove that*

$$\sin 2x = 2\sin x\cos x \quad \textit{and}$$

$$\cos 2x = \cos^2 x - \sin^2 x.$$

Problem 14 *Prove that*

$$\sin 3x = 3 \sin x - 4 \sin^3 x \quad \textit{and}$$

$$\cos 3x = 4 \cos^3 x - 3 \cos x.$$

A complex number z is called an *n-th root of unity* if $z^n = 1$ for some positive integer n.

Problem 15 *Use the formula* $e^{i2\pi} = 1$ *proven in Problem 8 to find all the cubic roots of unity in both polar and rectangular form. How are they related? Draw all the cubic roots of unity as vectors on one picture.*

Lemma 1

$$\sqrt[n]{1} = e^{i\,2\pi k/n} \quad where \quad k = 0, 1, 2, \ldots, n-1. \tag{7}$$

Problem 16 *Prove Lemma 1.*

Problem 17 *Draw all the 6-th roots of unity as vectors on one picture.*

For a depressed cubic equation $x^3 + px + q = 0$, let

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \text{ and } u = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

Further, let

$$\zeta = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

be a cubic root of unity.

Theorem 1 *The following are three different roots of the depressed cubic equation $x^3 + px + q = 0$.*

$$x_1 = u + v, \qquad x_2 = \zeta u + \zeta^2 v, \quad \textit{and} \quad x_3 = \zeta^2 u + \zeta v.$$

Problem 18 *Prove Theorem 1.*

Problem 19 *Find all the three roots of the cubic equation*

$$z^3 + 6z^2 + 12z + 18 = 0.$$

Now we are ready to get back to Problem 9 of the previous handout. In Problem 8 of the handout, we have found by elementary (not involving the Cardano formula) means that the roots of the equation

$$x^3 - 5x - 2 = 0$$

are $x_1 = -2$, $x_2 = 1 + \sqrt{2}$, and $x_3 = 1 - \sqrt{2}$. The Cardano formula, on the other hand, gives us the root

$$\sqrt[3]{1 + i\,\frac{7}{3}\sqrt{\frac{2}{3}}} + \sqrt[3]{1 - i\,\frac{7}{3}\sqrt{\frac{2}{3}}}. \tag{8}$$

It will be the goal of our investigation to figure out which of the roots x_1, x_2, x_3 formula (8) stands for.

Problem 20 *Let*

$$u = \sqrt[3]{1 + i\,\frac{7}{3}\sqrt{\frac{2}{3}}}.$$

and let φ be the angle the vector u forms with the real axis. Please find the following.

- $u^3 =$
- $v^3 =$
- $|u^3| =$
- $|v^3| =$
- $\sin 3\varphi =$
- $\sin(-3\varphi) =$

- $\cos 3\varphi =$
- $\cos(-3\varphi) =$

Problem 21 *Discuss the picture below with the students at your table.*