ORMC: GAME THEORY

OLYMPIAD GROUP 1, WEEKS 5

Problem 1. (Problems with stones). On a table there are n stones. Players A and B play a game where they alternately take stones from the table. At each step, the player to move will take a number s of stones, where s belongs to some fixed set S that A and B have decided on before. Player A always starts the game, and the player who cannot make their move loses. Determine which player wins for each n (assuming perfect play), in the following situations:

(a) $S = \{1\}$ (Yes, this is really easy)
(b) $S = \{1, 2\}$
(c) $S = \{1, 11, 21, 31, \ldots\}$
(d) $S = \{\text{powers of } 2\}$
(e) $S = \{3, 5, 8\}$
(f) $S = \{1, 2, 3\} \cup T$, where T has only odd numbers.
(g) Show that if S is finite, then the pattern eventually repeats.

Problem 2. (Stones but in 2D).

(a) Suppose we are given two piles with m and n stones respectively. Players A and B play a game where they alternatively take any number of stones from exactly one of the two piles (they're not allowed to take stones from both piles on the same turn). Player A starts the game, and the player who cannot make their move loses. For what values of m and n does player B win (assuming perfect play)?

(b) Same problem, but both A and B can take at most 3 stones from a pile of their choice at each turn.

Problem 3.

(a) We have a perfectly circular table. Players A and B take turns alternately, where the player to move places a quarter on the table. The rule is that quarters should never overlap, and should not hang outside the table. If A begins, who has the winning strategy?

(b) Consider the same game, but with three players A, B, C who take turns circularly in this order (A starts). Show that A and C can team up to ensure that B loses.

Problem 4. The number 2020 is written on the board. Players A and B play a game where they take turns alternatively; at each turn, a player can erase one number n from the board and replace it with either of the following:

(i). Two positive integers a, b such that $a + b = n$, or
(ii). Two integers $a, b \geqslant 2$ such that $ab = n$.

Show that the game must eventually end (the player who can't make any more moves loses), and determine which player has a winning strategy (if player A starts).

Problem 5.

(a) We are given a 1×2021 board (i.e., 2021 empty slots in a line). At each turn, players A and B (who take turns alternatively) can draw O's in three consecutive slots of the board, provided that all of them are empty before this move. Player A starts, and the player who can't make a move loses. Determine which player has a winning strategy.

(b) Same problem, but with a *circular* 1×2020 board.

Problem 6. A regular n-gon is drawn on the board. Players A, B and C take turns circularly in this order, and player A starts. At each turn, a player must connect two vertices of the n-gon with a line segment, provided that this line segment wasn't already drawn and that it doesn't cross another line segment already drawn (common vertices are okay). As before, the player who can't make a move loses; determine for which values of n player A wins.

Problem *7. (If you know how to play chess) Consider this position. Show that if it's white's turn to move, then black can force a draw, and that if it's black's turn to move, then white can force a win. What changes if we move everything up one square?

Homework

Problem 1. Like problem 1 from above, but with $S = \{1, 2, \dots, 2020\}$.

Problem 2. Like problem 2 from above, but the player to move can either take exactly one stone, or two stones from different piles; so

$$(-1, 0) \text{ or } (0, -1) \text{ or } (-1, -1).$$

Problem 3. We have a $2 \times n$ board, and Players A and B take turns in placing dominoes on the board. The rule is that no two dominoes are allowed to overlap, and the dominoes need to line up perfectly with the tiles. Player A moves first, and the player who can't move loses. Who has the winning strategy, depending on n? *Hint: do the $n = 2k$ case first.*