PRIME GENERATING FUNCTIONS II

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ADVANCED 2
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Last week, we explored the limitations of polynomials as prime generating functions. This week, we will try to define prime generating functions with recursive formulas.

Recursive Formulas that Are Basically No Help

Definition 1. Let $a(n) = a(n-1) + \gcd(n, a(n-1))$ with $a(1) = 7$. The prime generating function in question is $g(n) = a(n) - a(n-1) = \gcd(n, a(n-1))$ for $n \geq 2$.

Problem 1.
(a) Compute $a(1)$ through $a(11)$. When $\gcd(n, a(n-1))$ is prime, can you predict the next value of $a(n)$?
(b) Compute $g(2)$ through $g(11)$.

We claim that $g(n)$ is either 1 or prime. We will prove a specific result in Problem 2 that will allow us to prove the claim in Problem 3.

Problem 2. In this problem, take n_1 satisfying $2n_1 \geq 3$ and $a(n_1) = 3n_1$. Let n_2 be the next input where $g(n_2) \neq 1$. We will show that $g(n_2)$ is prime. Let k denote $n_2 - n_1$ and p denote the smallest prime divisor of $2n_1 - 1$. Since $2n_1 - 1$ is odd, p is odd.
(a) Let $1 \leq i \leq k$. Show that $g(n_1 + i) = \gcd(n_1 + i, 3n_1 + i - 1)$.
(b) Use part (a) to show that $g(n_1 + i)$ divides $2n_1 - 1$ and $2i + 1$.
(c) Prove that $k \geq \frac{p-1}{2}$.
(d) Prove that $k \leq \frac{p-1}{2}$. Conclude that $k = \frac{p-1}{2}$.
(e) Show that $g(n_1 + \frac{p-1}{2})$ divides the prime p. Conclude that $g(n_2) = p$.
(f) Show that $a(n_2) = 3n_2$.

Problem 3. Use Problem 2 to prove that $g(n)$ is always either 1 or prime for $a(1) = 7$. (Hint: We need to choose n_1 satisfying $2n_1 \geq 3$. Then show that n_2 satisfies the conditions to continue the process.)

Problem 4. Will this sequence contain all prime numbers eventually? If not, find a prime that is missing.

It is not yet known whether g generates all odd prime numbers.

There is another recursively-defined prime generating function that does produce all the prime numbers in a predictable way. However, as we will see, the definition requires us to know the prime numbers ahead of time.

Definition 2. Let $f(n) = \lfloor f(n-1) \rfloor (f(n-1) - \lfloor f(n-1) \rfloor + 1)$. The sequence $\lfloor f(n) \rfloor$ will enumerate the primes. Let p_n denote the nth prime number and P_n the product of the primes less than p_n. We define the initial condition as the convergent infinite series $f(1) = \sum_{n=1}^{N} \frac{p_n - 1}{P_n}$ for some integer N.

We will show that the choice of N determines how many primes the sequence enumerates. If you are familiar with infinite series, taking $f(1) = \sum_{n=1}^{\infty} \frac{p_n - 1}{P_n}$ will enumerate all the prime numbers in increasing order.

Problem 5. Let $f(1)$ be defined only by the first three terms in the sequence so $f(1) = \frac{2-1}{1} + \frac{3-1}{2} + \frac{5-1}{2\cdot 3} = \frac{8}{3}$. Show that $\lfloor f(n) \rfloor$ enumerates the primes for $1 \leq n \leq 2$.

Problem 6. Let $f(1)$ be defined only by the first four terms in the sequence so $f(1) = \frac{2-1}{1} + \frac{3-1}{2} + \frac{5-1}{2\cdot 3} + \frac{7-1}{2\cdot 3\cdot 5} = \frac{43}{15}$. Show that $\lfloor f(n) \rfloor$ enumerates the primes for $1 \leq n \leq 3$.

Problem 7. Let $f(1)$ be defined now by the first five terms in the sequence so $f(1) = \frac{2-1}{1} + \frac{3-1}{2} + \frac{5-1}{2\cdot 3} + \frac{7-1}{2\cdot 3\cdot 5} + \frac{11-1}{2\cdot 3\cdot 5\cdot 7} = \frac{306}{105}$. Show that $\lfloor f(n) \rfloor$ enumerates the primes for $1 \leq n \leq 4$.

In order to show that this sequence works as predicted, we will need to introduce an important result about the distribution of primes.

Theorem 1 (Bertrand's Postulate)**.** For each $n > 1$, there is a prime p such that $n < p < 2n$. In particular, $p_{i+1} < 2p_i$ and $p_{i+1} < \prod_{j=k}^{i} p_j$ for any $1 \leq k \leq i$.

Problem 8.

(a) Using induction, show $f(n) = p_n + \frac{(p_{n+1}-1)-p_n}{p_n} + \sum_{i=n+2}^{N} \frac{p_i - 1}{\prod_{j=n}^{i-1} p_j}$ for $1 \leq n \leq N-1$. For the sake of simplicity, we assume that the sum of the fractions is less than 1. Thus $\lfloor f(n) \rfloor = p_n$ for all $1 \leq n \leq N-1$. (Hint: You will need to apply Bertrand's Postulate.)

(b) Show that $f(N+i) = p_N - 1$ for all integers $i \geq 0$.

Problem 9. Discuss an enormous limitation of f as a formula for finding prime numbers.

Problem 10. Discuss other limitations of recursive formulas in general.