Problem 1 Find the coefficients $a$, $b$, $c$, and $d$ of the cubic function $y = ax^3 + bx^2 + cx + d$ given by the following graph.
A generic cubic equation

\[ az^3 + bz^2 + cz + d = 0 \]  
(1)

with real coefficients \( a \neq 0 \), \( b \), \( c \), and \( d \) is equivalent to the equation

\[ z^3 + \frac{b}{a}z^2 + \frac{c}{a}z + \frac{d}{a} = 0. \]  
(2)

**Problem 2** Find the variable change \( z = x - x_0 \) reducing the equation (??) to the depressed cubic form

\[ x^3 + px + q = 0. \]  
(3)
Problem 3 Use the expansion

\[(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = 3uv(u + v) + u^3 + v^3 \quad (4)\]

and the Vieta formulas for a quadratic equation to find a root of (??). Hint: rewrite (??) as \[x^3 = -px - q\] and compare the latter to (??).

The above method of solving a cubic equation was discovered by an Italian mathematician Scipione del Ferro (1465 - 1526), independently rediscovered by an Italian engineer Tartaglia (1500-1557), and published, with attribution to del Ferro, in 1545 by Gerolamo Cardano (1501-1576) in his book *Ars Magna*. In full accordance with Arnold’s Law, the formula for the root is known as the *Cardano formula*. 
Problem 4 Find a real root of the equation \( x^3 + 6x - 2 = 0 \).

Problem 5 Find a real root of the equation \( x^3 + 6x^2 + 9x - 2 = 0 \).
Long division of polynomials

**Example 1**   \[ \text{Divide } x^3 + 3x^2 + 5x - 4 \text{ by } x - 1. \]

**Step 1:** multiply \( x - 1 \) by a monomial of the form \( ax^n \) so that the leading term of the product equals the leading term of \( x^3 + 3x^2 + 5x - 4 \). In our case, \( ax^n = x^2 \). Subtract the product, \( x^2(x - 1) = x^3 - x^2 \), from the original polynomial.

\[
\begin{array}{c|ccccc}
 & x^2 & + 3x^2 & + 5x & - 4 \\
\hline
x - 1 & x^3 & \quad & \quad & \quad & \\
\hline
& x^3 & - x^2 & \quad & \quad & \\
\hline
& 4x^2 & + 5x & - 4 & \\
\end{array}
\]

**Step 2:** repeat step 1 for the polynomials \( 4x^2 + 5x - 4 \) and \( x - 1 \).

\[
\begin{array}{c|cccc}
 & x^2 & + 4x \\
\hline
x - 1 & x^3 & + 3x^2 & + 5x & - 4 \\
\hline
& x^3 & - x^2 & \quad & \quad & \\
\hline
& 4x^2 & + 5x & - 4 & \\
\hline
& -4x^2 & - 4x & \quad & \\
\hline
& 9x & - 4 & \\
\end{array}
\]

**Problem 6** Perform step 3 and finish the division in Example ?? above.
Problem 7  Divide $3x^5 - 2x^3 + 10x^2 - x + 11$ by $x^2 + x + 1$. 
Problem 8 Without using the Cardano formula, find all the three roots of the equation $x^3 - 5x - 2 = 0$. 
Problem 9 Use the Cardano formula to find a root of the equation $x^3 - 5x - 2 = 0$ from Problem ???. Which of the three roots found on the previous page is this one equal to? Hint: check them out one by one.
Complex numbers

The imaginary unit $i$ is defined as one of the two solutions of the equation $x^2 + 1 = 0$. The other one is $-i$.

Problem 10 Evaluate $i^{2017}$.

The set $\mathbb{C}$ of complex numbers is defined as the set of all the numbers of the form $z = a + ib$ where $a$ and $b$ are real numbers. The real part of the complex number $z$ is $\mathcal{R}(z) = a$. The imaginary part of the complex number $z$ is $\mathcal{I}(z) = b$. Complex numbers are constructed to have the same algebraic properties as real numbers. In particular, multiplication by $i$ is commutative for any real number, $ib = bi$. Addition of real and imaginary parts is also commutative, $a + ib = ib + a$.

Problem 11 Given $z = 1 - 3i + \sqrt{2}$, find $\mathcal{R}(z)$ and $\mathcal{I}(z)$.

Complex numbers were invented by Gerolamo Cardano in an attempt to resolve the paradox similar to the one we have encountered comparing the solutions of the cubic equation in Problems ?? and ?? . Complex numbers are very important. For example, complex numbers form the bedrock of quantum mechanics.
Problem 12  Prove that the sum of any two complex numbers, $v = a + ib$ and $w = c + id$, is also a complex number.

Note that addition of complex numbers is commutative, $v + w = w + v$.

Problem 13  Given $p = 7 - i$, $q = 2 + 2i$, and $r = -5 + i\sqrt{3}$, find $p + q + r$.

Problem 14  For the numbers $p$, $q$, and $r$ from Problem ??, find $p^2 + q^2 + r^2$.

Problem 15  Is any real number a complex number? Why or why not?
Problem 16 Prove that zero is the only neutral element with respect to addition of complex numbers. In other words, $z + n = z$ for any complex number $z$ and some complex number $n$ if and only if $n = 0$.

Problem 17 Prove that for any complex number $z$ there exists the opposite complex number, $-z$, such that $z + (-z) = 0$. What are $\mathcal{R}(-z)$ and $\mathcal{I}(-z)$?

Problem 18 Prove that multiplication of complex numbers is commutative. In other words, prove that for any $v = a + ib$ and $w = c + id$, $vw = wv$. What are $\mathcal{R}(vw)$ and $\mathcal{I}(vw)$?
Problem 19  For the numbers $p = 7 - i$, $q = 2 + 2i$, and $r = -5 + i\sqrt{3}$ from Problem ??, find the number $pqr$.

Problem 20  Prove that $1$ is the only neutral element with respect to multiplication of complex numbers. In other words, $zn = z$ for any complex number $z$ and some complex number $n$ if and only if $n = 1$. 
Problem 21  Prove that for any complex number $z = a + ib \neq 0$ there exists the inverse complex number, $z^{-1}$, such that $z^{-1}z = 1$. Hint: $a + ib \neq 0 \iff a^2 + b^2 \neq 0$. What are $\mathcal{R}(z^{-1})$ and $\mathcal{I}(z^{-1})$?

Problem 22  For the numbers $p = 7 - i$, $q = 2 + 2i$, and $r = -5 + i\sqrt{3}$ from Problem ??, find the numbers $p^{-1}$, $q^{-1}$, and $r^{-1}$. 
Let $z = a + ib$. The number $\bar{z} = a - ib$ is called \textit{conjugate} to $z$.

\textbf{Problem 23} For any two complex numbers $v$ and $w$, prove that $\bar{v + w} = \bar{v} + \bar{w}$.

\textbf{Problem 24} For any two complex numbers $v$ and $w$, prove that $vw = \bar{v} \bar{w}$.

\textbf{Problem 25} Prove that $z \in \mathbb{R} \iff z = \bar{z}$.

\textbf{Problem 26} Prove that for $z = a + ib$, $z\bar{z} = a^2 + b^2$.

Problem ?? justifies the following definition.

$$|z| = \sqrt{z\bar{z}} \quad (5)$$
Problem 27 For the numbers $p = 7 - i$, $q = 2 + 2i$, and $r = -5 + i\sqrt{3}$ from Problem ??, find $|p|$, $|q|$, and $|r|$.

Problem 28 Find all the (complex) roots of the equation $x^2 + 2x + 3 = 0$. 