Exploring Manifolds

1 Metrics

Recall the definition of a metric:

Definition: A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ with the following properties:

1. For any $x, y \in X$, $d(x, y) \geq 0$, with $d(x, y) = 0 \iff x = y$.
2. For any $x, y \in X$, $d(x, y) = d(y, x)$.
3. (Triangle Inequality) For all $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Problem 1.1: Why is the last property called the triangle inequality?

Problem 1.2: (Optional) Let X be a nonempty set, and define the *discrete metric* $d : X \times X \to \mathbb{R}$ given by $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$ otherwise. Verify that the discrete metric is a metric on X.

Problem 1.3: What is the usual (Euclidean) notion of distance on the number line $\mathbb{R}$? Write down a formula for $d(x, y)$, the distance between $x, y \in \mathbb{R}$. Prove that d is a metric.

Problem 1.4:

a) What is the usual (Euclidean) notion of distance in $\mathbb{R}^n$? Write down a formula for $d(x, y)$, where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, ..., y_n) \in \mathbb{R}^n$.

b) *(Optional Challenge)* Show that the function d defined in part a is indeed a metric.

Problem 1.5: What metric should we put on the set of complex numbers, $\mathbb{C}$? What about $\mathbb{C}^n$?

Definition: A *metric space* (X, d) is a set X along with a metric $d : X \times X \to \mathbb{R}$.

Problem 1.6: Prove or disprove: Let (X, d_X) be a metric space and suppose $Y \subset X$. Then Y can be turned into a metric space as well, with some metric d_Y which is consistent with the metric d_X (i.e. for all $a, b \in Y$, $d_Y(a, b) = d_X(a, b)$).

Problem 1.7: Show that the unit circle, $S^1 = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$, and the unit sphere, $S^2 = \{(a, b, c) \in \mathbb{R}^2 : a^2 + b^2 + c^2 = 1\}$, are metric spaces. In fact, show $S^n := \{(a_1, ..., a_n) \in \mathbb{R}^n : \sum_{i=1}^n a_i^2 = 1\}$ is a metric space.

Definition: Let (X, d) be a metric space. An *open ball* centered at $x \in X$ with radius $r > 0$ is the set $B(x, r) = \{y \in X : d(x, y) < r\}$.

Problem 1.8:

a) In the metric space $\mathbb{R}$ with the Euclidean metric (see problem 1.3), draw the open ball $B(0, 1)$. What does the open ball $B(x, r)$ look like in general?

b) In the metric space $\mathbb{R}^2$ with the Euclidean metric (see problem 1.4), draw the open ball $B((0, 0), 1)$. What does the open ball $B((x, y), r)$ look like in general?

c) Show the open unit square $(0, 1) \times (0, 1) \subset \mathbb{R}^2$ is a union of open balls.

   Hint: Let $(x, y) \in (0, 1) \times (0, 1)$. Find a possible radius for a small open ball centered at (x, y) that fits entirely inside the unit square.

Definition: Let (X, d_X) and (Y, d_Y) be metric spaces. We say $f : X \to Y$ is *continuous* if for every $y \in Y$ and $r > 0$, we have the set $\{x \in X : f(x) \in B(y, r)\}$ is the union of open balls in X.

Problem 1.9: Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$. Show that $f(x)$ is continuous by showing that for any open interval $(a, b) \subset \mathbb{R}$, the set $\{x \in \mathbb{R} : f(x) \in (a, b)\}$ is either itself an open interval, or the union of two open intervals.

Definition: We say two metric spaces (X, d_X) and (Y, d_Y) are *homeomorphic* we have a function $f : X \to Y$ which is **continuous**, **bijective**, and whose **inverse is also continuous**.

Problem 1.10: Show $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$.

2 Topological Manifolds

Now, we focus our attention to a special class of metric spaces.

Definition: A *topological n-manifold* (or *n-manifold*, or simply *manifold*) is a metric space (X, d) such that for each $x \in X$, we have some open ball $B(x, r) = \{y \in X : d(y, x) < r\}$ which is homeomorphic to $\mathbb{R}^n$. (It should be the same integer n for every point!) The *dimension* of the n-manifold X is simply the integer n.

Problem 2.1: You've probably seen some topological manifolds before. Argue why each of the following are indeed topological manifolds.

a) $\mathbb{R}^n$

b) $\mathbb{C}$

c) $\mathbb{C}^n$

d) The open interval $(0, 1) \subset \mathbb{R}$

e) The open unit square $(0, 1) \times (0, 1) \subset \mathbb{R}^2$

f) The unit circle $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$

g) The unit sphere $\{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\}$

Problem 2.2: Determine which of the following are manifolds (assigning each set the metric it inherits from $\mathbb{R}^n$ or $\mathbb{C}^n$ as appropriate). For those that are, what is their dimension? For those that are not, why not?

a) The closed unit square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$

b) The open unit disk $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 < 1\}$

c) The closed unit disk $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 1\}$

d) The x-axis $\{(a, b) \in \mathbb{R}^2 : b = 0\}$

e) The x-axis and y-axis, i.e. the set $\{(a, b) \in \mathbb{R}^2 : a = 0 \text{ or } b = 0\}$

f) The subset $\{z \in \mathbb{C} : |z| = 1\}$

g) The subset $\{(a, b) \in \mathbb{C}^2 : a^2 + b^2 = 1\}$

h) The subset $\{(a, b) \in \mathbb{C}^2 : a^2 + b^2 = 0\}$

i) The subset $S^n = \{(a_1, ..., a_n) \in \mathbb{R}^n : \sum_{i=1}^n a_i^2 = 1\}$

j) The set $\{(x, y, z) \in \mathbb{R}^3 : xz = 0 \text{ and } yz = 0\}$

3 Quotient Spaces

In this section, we discuss a useful way to describe more complicated spaces by means of ”gluing” parts of a different space together.

3.1 Equivalence Relations

Recall an *equivalence relation* on a set X is a relation $\sim$ between points in X which satisfies

1. (Reflexivity) For any $x \in X$, $x \sim x$.
2. (Symmetry) For any $x, y \in X$, if $x \sim y$, then $y \sim x$.
3. (Transitivity) For any $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

The equivalence class of an element $x \in X$, denoted $[x]$, is the set of all elements in X related to it. That is, $[x] = \{y \in X : x \sim y\}$.

We define the set $X/\sim$ to be the set of all equivalence classes under the equivalence relation $\sim$. That is, $X/\sim$ is the set $\{[a] : a \in X\}$.

Geometrically, we may visualize $X/\sim$ as taking X and gluing together all equivalent points.

Problem 3.1: Let $X = [0, 1] \times [0, 1]$ be the unit square. For each of the following equivalence relations, draw a picture to illustrate which points should be glued together to get $X/\sim$. What is the resulting space look like?

a) $(x, 0) \sim (x, 1)$ for each $x \in [0, 1]$.

b) $(x, 0) \sim (1 - x, 1)$ for each $x \in [0, 1]$.

c) $(x, 0) \sim (x, 1)$ for each $x \in [0, 1]$, and $(0, y) \sim (1, y)$ for each $y \in [0, 1]$.

d) $(x, 0) \sim (x, 1)$ for each $x \in [0, 1]$, and $(0, y) \sim (1, 1 - y)$ for each $y \in [0, 1]$

Problem 3.2: Explain how you might be able to create each of the above spaces using a square sheet of paper and some glue. Actually, for one of them, you might need some special paper that can pass through itself freely. Do you know which one that is? Do you recognize the space?

Problem 3.3: Which of the above spaces are manifolds? For each one, explain why or why not. (The rest are actually *manifolds with boundary*. Unfortunately, we won't think about those too much).

3.2 Quotient by a subspace

There is a special case of the above construction. In this, we take a set X and a subset $A \subset X$, and set up an equivalence as follows: all points in the subset A are considered equivalent. Instead of writing $X/\sim$ as above, we write X/A instead.

Geometrically, we visualize X/A as taking the space X and "crushing" all of A to a single point (or gluing all the points in A together).

Problem 3.4: Let $X = S^1$ and let $a, b \in X$ be two distinct points. Describe X/A, where $A = \{a, b\}$. Is it a manifold? Why or why not?

Problem 3.5: Let $X = [0,1]$ and let $A = \{0,1\} \subset [0,1]$. Describe X/A. What familiar space is it homeomorphic to? Is it a manifold?

Problem 3.6: Let $X = [0,1] \times [0,1]$ and $A = \{(x,y) \in [0,1] \times [0,1] : x = 0, x = 1, y = 0, \text{ or } y = 1\}$, i.e. the boundary of the unit square X. Describe X/A. What familiar space is it homeomorphic to? Is it a manifold?

Hint: It may help to try constructing this with a piece of paper.

Exploring Projective Space

In this section, we will construct the manifold $\mathbb{RP}^n$, which will be an extension of $\mathbb{R}^n$ known as *real projective space*. We will think of various ways to visualize it.

4 Lines through $\mathbb{R}^2$

Problem 4.1:

a) Define $\sim$ on $\mathbb{R}^2$ by $(a,b) \sim (c,d)$ if and only if $ad = bc$. Is $\sim$ an equivalence relation?

b) Define $\sim'$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ by $(a,b) \sim' (c,d)$ if and only if $ad = bc$. Is $\sim'$ an equivalence relation?

c) Define an equivalence relation $\sim''$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ with $(a,b) \sim (\lambda a, \lambda b)$ for each $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $\lambda \neq 0$. Show $\sim'$ and $\sim''$ are the same relation.

d) What are the equivalence classes of the relation $\sim'$ (or equivalently, of $\sim''$)?

Definition: Define $\mathbb{RP}^1$ as the space $(\mathbb{R}^2 \setminus \{0\}) / \sim''$.

That probably isn't the most enlightening definition, so let's be more explicit:

$\mathbb{RP}^1$ consists of equivalence classes of points (a,b), where a, b are not both 0. The equivalence class of (a,b) is denoted $[a:b]$. We write

$$\mathbb{RP}^1 = \{[a:b] \mid a \neq 0 \text{ or } b \neq 0\}$$

Problem 4.2: Which points of $\mathbb{R}^2$ are in $[2:3]$? What about $[0:2]$? $[7:0]$? $[0:0]$?

Explicitly, we know

$$[a:b] = \{(\lambda a, \lambda b) \in \mathbb{R}^2 : \lambda \neq 0\}$$

Problem 4.3: Show $[2:3] = [8:12]$.

Observe that lines in $\mathbb{R}^2$ can be classified in a nice way: by their slope!

Problem 4.4:

a) Show every element of $\mathbb{RP}^1$ except for one can be written uniquely as $[1:m]$, where m is the slope of the corresponding line in $\mathbb{R}^2$.

b) Which line in $\mathbb{R}^2$ does the remaining element of $\mathbb{RP}^1$ correspond to?

c) Consider the lines $[1:m]$ as m goes to $+\infty$. Which line do we approach?

d) Consider the lines $[1:m]$ as m goes to $-\infty$. Which line do we approach?

e) (Open-ended) Is there a sensible 'slope' we could give the line in part b?

5 Visualization of $\mathbb{RP}^1$

With the previous problem in mind, we observe the following: the elements of $\mathbb{RP}^1$ correspond to elements of $\mathbb{R}$, except that $\mathbb{RP}^1$ has one extra point: a "point at infinity". This point somehow sits *simultaneously* at $+\infty$ and $-\infty$.

Problem 5.1: Write down an injective function $f : \mathbb{R} \to \mathbb{RP}^1$ whose image consists of every point except for $[0 : 1]$. Use this to explain why we might visualize $\mathbb{RP}^1 \setminus \{[0 : 1]\}$ to look the same as $\mathbb{R}$.

Problem 5.2: Now we attempt our first visualization of $\mathbb{RP}^1$.

a) Draw $\mathbb{RP}^1 \setminus \{[0 : 1]\}$ as a number line. What does each point on the number line represent?

b) Where should we add $[0 : 1]$ to complete our picture? *Hint*: See $4.4c - 4.4e$.

c) What does $\mathbb{RP}^1$ look like? *Hint*: Compare with problem 3.5.

We'll call this visualization of $\mathbb{RP}^1$ the "number line with a point at infinity".

Oddly, $\mathbb{RP}^1$ looks like S^1. In a way, though, it's actually closer to "half" of our usual unit circle, as the following problem shows.

Problem 5.3: In this problem, we try to visualize $\mathbb{RP}^1$ without leaving our starting space of $\mathbb{R}^2 \setminus \{(0, 0)\}$, which we already know how to visualize.

a) Since each element of $\mathbb{RP}^1$ corresponds to one line (through the origin) in $\mathbb{R}^2$, let's pick a representative element for each line. Show that any element of $\mathbb{RP}^1$ can be written as $[a : b]$, where $a^2 + b^2 = 1$. Is this representation unique?

b) Show that we can even write each element as $[a : b]$ for $a^2 + b^2 = 1$ and $a \geq 0$.

c) Show that there is one line (through the origin) in $\mathbb{R}^2$ with two representatives satisfying the conditions in part b, but the rest of the lines have precisely one representative.

d) Conclude that $\mathbb{RP}^1$ may be visualized as a semicircle with its two endpoints glued together. What does that space look like?

In this case, we come to the same conclusion: $\mathbb{RP}^1$ still looks like S^1, but is different from at least some perspective.

Problem 5.4: Show there is a surjective map $f : S^1 \to \mathbb{RP}^1$ such that every point in $\mathbb{RP}^1$ is mapped to by precisely two points in S^1. (We say S^1 is a "double cover" of $\mathbb{RP}^1$.)

We close with trying to understand what happens to lines *not* going through the origin in $\mathbb{R}^2$.

Problem 5.5: Let L be the line $L = \{(x, y) \in \mathbb{R}^2 : y = x + 1\} \subset \mathbb{R}^2$.

a) The point $(5, 6) \in L$ corresponds to $[5 : 6] = [1 : 6/5]$ in $\mathbb{RP}^1$, which corresponds to the point $6/5$ on our number line with a point at infinity (see 5.2). Consider the sequence of points $(1, 2)$, $(2, 3)$, $(3, 4)$, $\ldots \in L$. Sketch the corresponding points in our number line.

b) Repeat this process for the points $(-1, 0), (-2, -1), (-3, -2), \ldots \in L$.

c) Where does $(0, 1) \in L$ end up?

d) What's the only point in $\mathbb{RP}^1$ we are missing? Why?

6 Lines through $\mathbb{R}^3$

In a similar setup as before, we put an equivalence relation on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ via $(a,b,c) \sim (\lambda a, \lambda b, \lambda c)$ for all $(a,b,c) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ and $\lambda \neq 0$. We denote the equivalence class of (a,b,c) as $[a:b:c]$. It consists of nonzero points in the line in $\mathbb{R}^3$ going through the origin and (a,b,c). That is

$$[a:b:c] = \{(\lambda a, \lambda b, \lambda c) \in \mathbb{R}^3 : \lambda \neq 0\}$$

Finally, we define

$$\mathbb{RP}^2 = \{[a:b:c] \mid a \neq 0 \text{ or } b \neq 0 \text{ or } c \neq 0\}$$

Problem 6.1:

a) Which points of $\mathbb{R}^3$ are in $[1:2:3]$? $[1:0:0]$?

b) Find another way to write $[4:5:6]$. That is, find a different representative element of that line.

Problem 6.2:

a) Show that every element of $\mathbb{RP}^3$ can be written as $[a:b:c]$ with $a^2+b^2+c^2=1$ and $a \geq 0$.

b) Which points do not have a unique representative element satisfying the above properties?

c) Show the elements that do have a unique representative are in bijection with $\mathbb{R}^2$.

d) Show the elements that do not have a unique representative are in bijection with $\mathbb{RP}^1$.

e) Conclude $\mathbb{RP}^2$ can be thought of as $\mathbb{R}^2$ along with a copy of $\mathbb{RP}^1$. Just as we thought of $\mathbb{RP}^1$ as a number line ($\mathbb{R}^1$) with an extra point, we will think of $\mathbb{RP}^2$ as the plane ($\mathbb{R}^2$) with many extra points.

For $\mathbb{RP}^1$, we decided it corresponded to a number line, with the extra element $[0:1]$ placed simultaneously at $+\infty$ and $-\infty$. Where do we put all of our extra points in the case of $\mathbb{RP}^2$?

Problem 6.3: The elements of the form $[1:a:b] \in \mathbb{RP}^2$ can be visualized as the plane (with $[1:a:b]$ corresponding to (a,b)). Here, we show $[0:1:\lambda]$ can be visualized as the point at infinity for the line $L = \{(x,y) \in \mathbb{R}^2 : y = \lambda x\}$, and that $[0:0:1]$ is the point at infinity for the y-axis.

a) The elements $[1:x:2x] \in \mathbb{RP}^2$ correspond to points on the line $y = 2x$ in our visualization. For $x \neq 0$, we may rewrite this as $[1:x:2x] = [\frac{1}{x}:1:2]$. What would be a reasonable interpretation of $[0:1:2]$?

b) Generalize the above argument for the elements $[1:x:\lambda x] \in \mathbb{RP}^2$ (i.e. the line $y = \lambda x$ in our visualization), as well as for the points $[1:0:y] \in \mathbb{RP}^2$ (i.e. the y-axis in our visualization).

So, $\mathbb{RP}^2$ can be visualized (or even constructed) as follows:

1. Start with $\mathbb{R}^2$.
2. For each line through the origin, add a point at infinity. That is, add a single point that is simultaneously at the $+\infty$ and $-\infty$ ends of the line.

And voila! We have $\mathbb{RP}^2$. Yet, it's still not super clear what it looks like. Is it ike S^2? Note that S^2 can be constructed by the following process:

1. Start with $\mathbb{R}^2$.
2. Add a single point, which simultaneously sits at $+\infty$ and $-\infty$ ends of *every* line at the same time.

So at least from the construction, $\mathbb{RP}^2$ is not S^2. Yet, we have an analogous result as in 5.4.

Problem 6.4: Show there is a surjective map $f : S^2 \to \mathbb{RP}^2$ such that every point in $\mathbb{RP}^2$ is mapped to by precisely two points in S^2. (We say S^2 is a double cover of $\mathbb{RP}^2$.)

As it turns out, S^2 has no connected double covers, so if we accept this fact, we have $\mathbb{RP}^2$ and S^2 are not the same.

We once again close this section by trying to understand lines in $\mathbb{R}^2$ not going through the origin, this time looking in $\mathbb{RP}^2$ instead.

Problem 6.5: Let $L = \{(x, y) \in \mathbb{R}^2 : y = x + 1\}$.

a) Which elements of $\mathbb{RP}^2$ can be visualized as the points $(x, x + 1)$ in $\mathbb{R}^2$?

b) For $x \neq 0$, we have $[1 : x : (x + 1)] = [\frac{1}{x} : 1 : 1 + \frac{1}{x}]$. What would be a suitable "point at infinity" for the line L?

c) Repeat this process for $y = x$ and $y = x + 2$.

d) Show that any two parallel lines share the same point at infinity (in $\mathbb{RP}^2$).

7 $\mathbb{RP}^n$ is a manifold

As it turns out, our constructions of $\mathbb{RP}^1$ and $\mathbb{RP}^2$ are manifolds. We partially prove it below.

Problem 7.1: Show there is an injective funtion $f : \mathbb{RP}^1 \to \mathbb{R}^2$. (You do not need to write this function out explicitly, though you may).

In this way, $\mathbb{RP}^1$ inherits a metric.

Problem 7.2: (Challenge) Show there is an injective function $f : \mathbb{RP}^2 \to \mathbb{R}^4$. *Hint*: Start with $g : \mathbb{R}^3 \to \mathbb{R}^4$ given by $g(x, y, z) = (xy, xz, yz, x^2 - y^2)$.

In this way, $\mathbb{RP}^2$ inherits a metric.

This addresses the "metric space" aspect of being a manifold. Yet, we also want each point in $\mathbb{RP}^1$ to have an open ball that looks like $\mathbb{R}$, and each point in $\mathbb{RP}^2$ to have an open ball that looks like $\mathbb{R}^2$. Do we?

Problem 7.3: Show that there is an injective function $f : \mathbb{R}^1 \to \mathbb{RP}^1$ whose image contains every point except $[0 : 1]$. Similarly, show there is an injective function $g : \mathbb{R}^1 \to \mathbb{RP}^1$ whose image *does* contain $[0 : 1]$ (but doesn't contain, say, $[1 : 0]$).

In this sense, every point in $\mathbb{RP}^1$ at least has a subset which "looks like" $\mathbb{R}$ in our informal visualization of $\mathbb{RP}^1$. One can formalize this (though we won't) to conclude $\mathbb{RP}^1$ is a 1-dimensional manifold.

Problem 7.4: Let $[a : b : c] \in \mathbb{RP}^2$ be arbitrary. Show there is an injective function $f : \mathbb{R}^2 \to \mathbb{RP}^2$ with $[a : b : c]$ in its image.

When formalized, this too shows $\mathbb{RP}^2$ is a 2-dimensional manifold.

Problem 7.5: (Challenge) Let's be more formal. Define a metric on $\mathbb{RP}^2$ as $d([a:b:c],[x:y:z])$ as follows: it will be the minimal distance between a point in $[a:b:c] \cap S^2 \subset \mathbb{R}^3$ and $[x:y:z] \cap S^2 \subset \mathbb{R}^3$. That is, take the line corresponding to $[a:b:c]$ and find the two points $A_1, A_2 \in \mathbb{R}^3$ which lie on the line and the sphere. Similarly, find the two points $B_1, B_2 \in \mathbb{R}^3$ for $[x:y:z]$. Then, set the distance between $[a:b:c]$ and $[x:y:z]$ to be the minimum of the four distances $|A_i - B_j|$.

a) Show that d is a metric.

b) Show that every element in $\mathbb{RP}^2$ has an open ball (under this metric) which is homeomorphic to $\mathbb{R}^2$.

8 Other Projective Spaces (Bonus Section)

Problem 8.1: Define $\mathbb{RP}^n$. Show that it contains a copy of $\mathbb{R}^n$ as well as some extra points, forming a copy of $\mathbb{RP}^{n-1}$. How would you visualize $\mathbb{RP}^n$?

Problem 8.2: Define $\mathbb{CP}^1$. Show that it is homeomorphic to S^2.

9 More Paper Constructions (Bonus Section)

Problem 9.1: Let $X = [0,1] \times [0,1]$ be the unit square. Consider the equivalence relation $\sim$ with $(0,y) \sim (y,0)$ and $(x,1) \sim (1,x)$ for all $x, y \in \mathbb{R}$. Describe $X/\sim$. Do you recognize the space?

Problem 9.2: Let $X = [0,1] \times [0,1]$ be the unit square. Consider the equivalence relation $\sim$ with $(0,y) \sim (1,1-y)$ and $(x,1) \sim (1-x,0)$ for all $x, y \in \mathbb{R}$. Describe $X/\sim$. Do you recognize the space?

Problem 9.3: In problem 3.1, we constructed a torus by taking the quotient of a square. Construct a two-holed torus by taking the quotient of an octagon.