ORMC Advanced 1A **2020**

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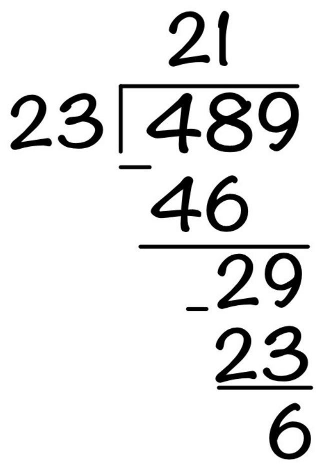
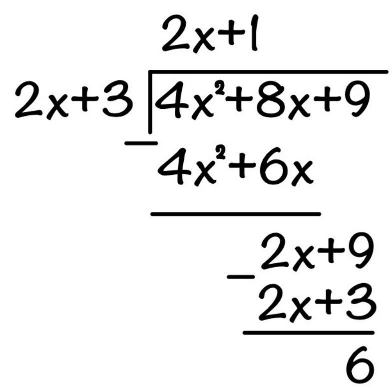
We can do with polynomials a lot of the operations we can do with numbers: addition, subtraction, multiplication. But can we divide them? Yes!





Unfortunately, division is not always so simple

Problem 1 *Divide* $4x^2 + 5x + 7$ *by* $2x + 3$.

Problem 2 *Divide* $x^3 - x^2 - x - 15$ *by* $x - 3$.

Problem 3 *Divide* $x^3 - x^2 - x - 15$ *by* $x^2 - 3$.

Prove the problem 1a from the previous class using long division:

Problem 4 (1a from previous class)

a *Let* $ax^2 + bx + c = 0$ *be a quadratic equation. Show that if it has two distinct real roots* x_0, x_1*, then*

$$ax^2 + bx + c = a(x - x_0)(x - x_1).$$

b *Show that a quadratic equation cannot have more than two distinct real roots.*

c *Now suppose that* $ax^2 + bx + c$ *has exactly one real root* x_0*. Show that* $ax^2 + bx + c = a(x - x_0)^2$*.*

The following is known as a *discriminant* of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.

$$D = b^2 - 4ac \tag{1}$$

Theorem 1 *If $D < 0$, then the quadratic equation $ax^2+bx+c = 0$ with real coefficients $a \neq 0$, b, and c has no real roots. If $D \geq 0$, then the following is the formula for the roots of the equation.*

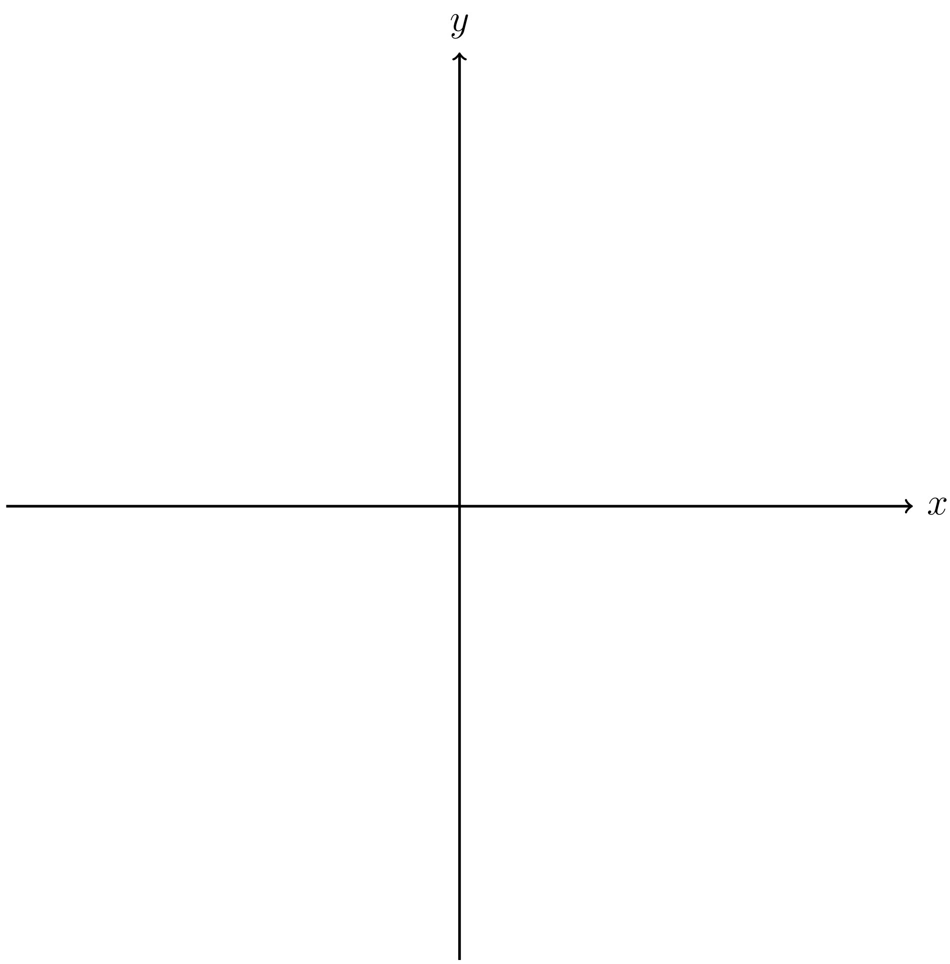
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \tag{2}$$

Problem 5 *Prove Theorem 1.*

Problem 6 *Find all the real solutions of the equation* $\sqrt{x-2} = x - 4$.

Problem 7 *Find all the real solutions of the equation* $7\left(x + \frac{1}{x}\right) - 2\left(x^2 + \frac{1}{x^2}\right) = 9$.

Problem 8 *Sketch the graph of the function $y = ax^2 + bx + c$, given the following information: $a > 0$, $b > 0$, $D < 0$.*



Is the coefficient c positive, negative, or zero? Why?

Problem 9 *Find all the real solutions of the equation* $2x^2 + 6 - 2\sqrt{2x^2 - 3x + 2} = 3x + 3.$

Problem 10 *Find all the real solutions of the equation* $\sqrt[3]{x+a} + \sqrt[3]{x+a+1} + \sqrt[3]{x+a+2} = 0.$

Vieta Formulas

Theorem 2 *Let* x_1 *and* x_2 *be the roots of the quadratic equation* $ax^2 + bx + c$, $a \neq 0$. *Then* $x_1 + x_2 = -b/a$ *and* $x_1 x_2 = c/a$.

Problem 11 *Prove Theorem 2.*

Problem 12 *Write down a quadratic equation that has the roots* $x_1 = 3$ *and* $x_2 = -4$.

Problem 13 *Generalize Vieta formulas to a cubic equation* $ax^3 + bx^2 + cx + d$, $a \neq 0$.

Problem 14 *Write down a cubic equation that has the roots* $x_1 = 1$, $x_2 = 2$, *and* $x_3 = 3$.

Problem 15 *Without solving the equation* $ax^2 + bx + c = 0$*, find the sum of the squares of its roots provided that* $a \neq 0$ *and* $D \geq 0$.

Problem 16 *Find all the prime numbers* p *and* q *such that the equation* $x^2 - px - q = 0$ *has a solution that is a prime number.*

A function $f(x)$ is called *convex* if for any x_1 and x_2 in its domain and for any $0 < \alpha < 1$,

$$f(\alpha x_1 + (1-\alpha)x_2) < \alpha f(x_1) + (1-\alpha)f(x_2). \tag{3}$$

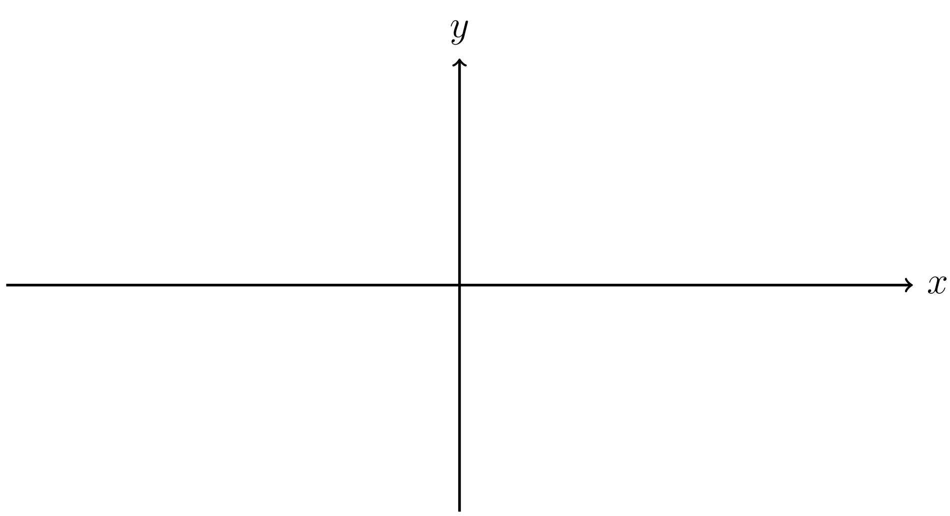
Problem 17 *Give a geometric interpretation to formula (3).*

Problem 18 *Prove that for a linear function* $f(x) = bx + c$, $f(\alpha x_1 + (1-\alpha)x_2) = \alpha f(x_1) + (1-\alpha)f(x_2)$ *for any value of the parameter* α.

Problem 19 *Prove that $f(x) = ax^2+bx+c$ is convex for $a > 0$.*

The value $\hat{x}$ is called a *minimum* of a function $f(x)$ if $f(\hat{x}) \leq f(x)$ for every x in the function's domain.

Problem 20 *Sketch the graph of a function having two minima.*



Problem 21 *The function $f(x)$ is convex. Prove that it can have at most one minimum.*

Problem 22 *Find the minimum of the function $f(x) = ax^2 + bx + c$, $a > 0$. Prove that it is indeed a minimum. What is the value of the function at the point?*

Problem 23 *Find the minimum of the function* $f(x) = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2$.

Problem 24 *A straight line in the plain is given by the equation* $ax + by + c = 0$. *Find the distance from the point* (x_0, y_0) *to the line.*

Problem 25 *Prove that for* $x > 0$, $x + \frac{1}{x} \geq 2$.

Problem 26 *Given* $x + y + z = 1$, $x > 0$, $y > 0$, *and* $z > 0$, *prove that*

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9.$$