

## Logic Puzzles: Wizards and Hats

UCLA Olga Radko Math Circle 10/18/2020

This week, we will be exploring some logic puzzles.

### Warm Up:

In the past, we visited the Island of Knights and Liars to solve some tricky puzzles. Let's revisit one of the puzzles we saw.

Remember, **Knights** always tells the truth and **Liars** always lie.

1. *Charlie, Donna, and Emily are islanders, not tourists. A tourist asked Charlie if he was a knight. Charlie mumbled something, chewing his gum, so the tourist did not understand his answer. The tourist asked Donna what Charlie said. Donna answered that Charlie claimed to be a liar. To that Emily responded, "Don't believe Donna, she is a liar herself!" What kinds are Charlie, Donna and Emily?*

Charlie: \_\_\_\_\_

Donna is a \_\_\_\_\_

Emily is a \_\_\_\_\_

We also learned about a very evil king who had a very unique way of punishing criminals. Law-breakers were given a choice between two doors. Behind each door, there was either a **hungry tiger** or a **treasure of gold**, but **not nothing** or **both**. The king would also post warnings on the doors and let the criminals choose.

2. *One day, there was a criminal facing the doors with the following signs.*

Door 1: *at least one of these rooms has gold*

Door 2: *a tiger is in the other room*

*"Are the signs true?" asked the prisoner. "They are either both true or both false," replied the king. Then he smiled warmly and added, "Make your choice, buddy!"*

*Which door should the prisoner open? Why?*

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### Warm Up Solutions:

1. Assume that Donna was telling the truth. Then Charlie had claimed to be a liar. This is impossible for either a knight or a liar. Therefore, Donna is a liar. Since Donna is a liar, Emily was telling the truth. Therefore, Emily is a knight. Since both knights and liars claim to be knights when asked what kind they are, Charlie could be either.
- 2.

**Problem 12.12:** the following table lists all the possible cases. The letter  $t$  stands for a tiger, the letter  $g$  stands for gold, and the letters  $T$  and  $F$  stand for the *True* and *False* values of the statements 1 and 2.

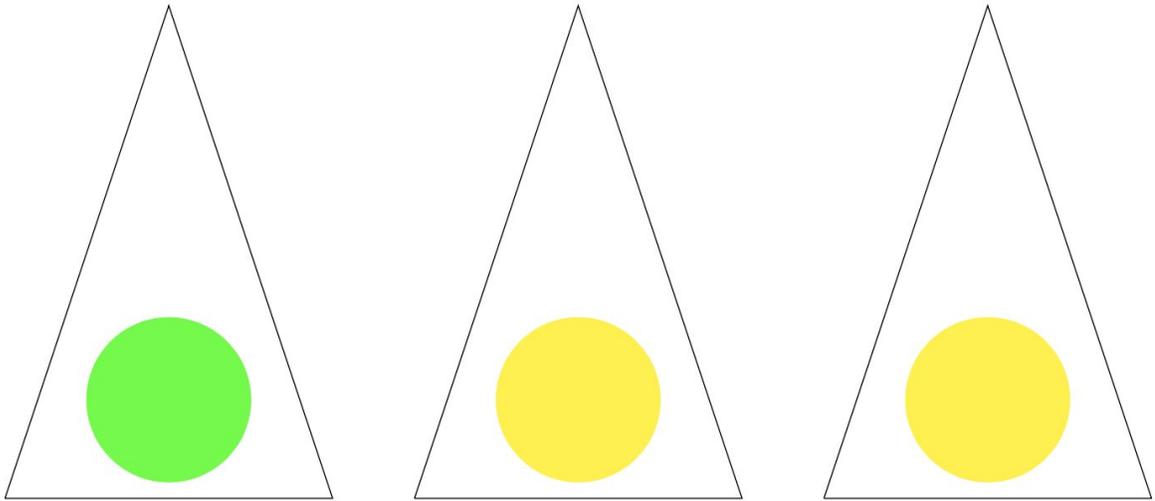
Door 1	Door 2	Statement 1	Statement 2
t	t	F	T
t	g	T	T
g	t	T	F
g	g	T	F

Do the following helping students to solve the problem.

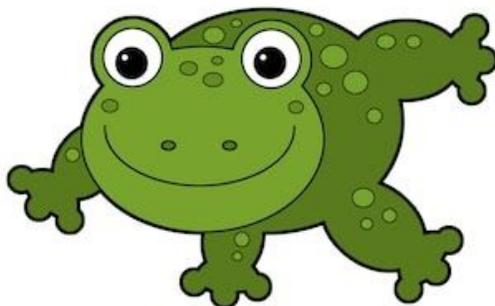
- Draw the header of the table on the board. Then ask students to list all the possible placings of tigers and gold behind the doors. This will fill the eight cells in the two columns on the left-hand side of the table.
- Ask students to evaluate whether each of the two statements is true or false in each of the four cases. This will fill the eight cells in the two columns on the right-hand side of the table.
- Point out to students that the two statements are never false at the same time.
- Both statements are true in the second case only. There is a tiger behind the first door and a treasure of gold behind the second one.

**Lesson:**

A part of wizardry training is developing logic skills. In the following exercises, two students will be presented with three wizard's hats, one with a green label and two with yellow labels.



The students will be asked to close their eyes. One hat will be hidden, the others will be put on the students' heads. The students will open their eyes, and then be asked, "Who knows what color hat he/she has on? If so, which color is it?" The students can figure out the color hat he/she is wearing only by observing the hat of the other student. **A student is not allowed to look at his/her own hat.** This kind of cheating will be punished by turning the cheat into a toad!



**Problem 1:** Kaley and Emmanuelle are given two hats. Kaley gets the green- labeled hat, and Emmanuelle gets a yellow-labeled one.

a) Cory, the lead wizard, asks them what color hat they have on. What are Kaley and Emmanuelle going to say? Explain.

Kaley: "I don't know"

Emmanuelle: "Yellow"

Kaley sees a yellow hat so she knows that Kaley either has a yellow or green hat. Emmanuelle sees a green hat and knows Emmanuelle must have a yellow hat.

Note: Assuming they reply at the same time

b) After hearing Emmanuelle's answer, can Kaley now figure out which color hat she has on? Explain why or why not.

Yes, Kaley can figure out Kaley has a green hat. If Kaley had a yellow hat, then Emmanuelle would not know if Emmanuelle had a green or yellow hat. However, since Emmanuelle knows Emmanuelle is yellow, that must mean Emmanuelle is looking at a green hat

**Problem 2:** This time, Kaley and Emmanuelle are both given the yellow- labeled hats.

a) Cory asks them what color hat they have on. What are Kaley and Emmanuelle going to say? Explain.

Kaley: "I don't know"

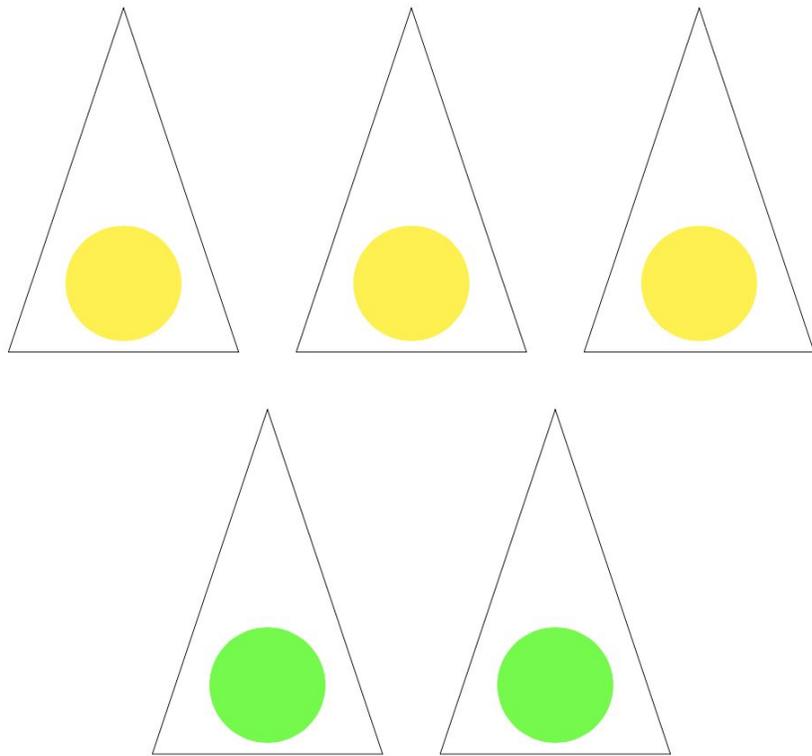
Emmanuelle: "I don't know"

They will both say they don't know because they each see a yellow hat and know the hat on their own head can be either yellow or green (again assuming they answer simultaneously).

b) Now that they heard each other's answers, can either of them figure out the color of their hat? How?

Yes. They both can guess they have yellow hats. This is because they each know the other doesn't know which hat they are wearing on their own head. This means they know the other is also looking at a yellow hat and can deduce they're both wearing a yellow hat.

As the training progresses, three students will be chosen and presented with five hats, three with yellow dots and two with green. Once again, the students will be asked to close their eyes. Two hats will be hidden, three will be put on the students' heads. The students will be asked to open their eyes and to figure out the color of their hat labels by observing those of the other students.



**Problem 3:** Kaley, Emmanuelle, and Lucy are given the hats. Kaley gets a yellow-labeled hat, Emmanuelle and Lucy get the hats with green labels.

a) *What will each person say when Cory asks them for the color of their hats? Explain.*

Kaley: \_\_\_\_\_ **“Yellow”**

Emmanuelle: \_\_\_\_\_ **“I don’t know”**

Lucy: \_\_\_\_\_ **“I don’t know”**

b) *After hearing each person’s answer, can others figure out which color hat they have on? How?*

Yes. Since Kaley knows she has a yellow hat, that means she must see two green hats. Hence, E and L will say they have a green hat.

**Problem 4:** This time, Kaley and Emmanuelle get yellow-labeled hats while Lucy gets the hat with a green label.

a) *How will each person respond to Cory’s question? Explain.*

Kaley: \_\_\_\_\_ **“I don’t know”**

Emmanuelle: \_\_\_\_\_ **“I don’t know”**

Lucy: \_\_\_\_\_ **“I don’t know”**

b) Now that they have heard each other's answers, who can say the color of their hat and who can't? Explain why or why not. (Hint: What would happen if one assumes that she is wearing the green hat?)

Kaley: \_\_\_\_\_ **"Yellow"**

Emmanuelle: \_\_\_\_\_ **"Yellow"**

Lucy: \_\_\_\_\_ **"I don't know"**

c) After hearing everybody's answers, who can now figure out the color of their hat? Explain. (Hint: What would happen if Lucy assumes she is wearing the yellow hat?)

Kaley: \_\_\_\_\_ **Figured it out in the previous round.**

Emmanuelle: \_\_\_\_\_ **Figured it out in the previous round.**

Lucy: \_\_\_\_\_ **Green**

**Problem 5:** Finally, all the three students are given the yellow-labeled hats.

a) How would each person respond to Cory's question? Explain.

Kaley: \_\_\_\_\_ **"I don't know"**

Emmanuelle: \_\_\_\_\_ **"I don't know"**

Lucy: \_\_\_\_\_ **"I don't know"**

b) After hearing everybody's answer, who can say the color of their hat and who can't? Explain why or why not. (Hint: What would happen if one of them assumes she is wearing a green hat?)

Kaley: \_\_\_\_\_ "I don't know"

Emmanuelle: \_\_\_\_\_ "I don't know"

Lucy: \_\_\_\_\_ "I don't know"

c) After hearing each person's answer, Lucy tries to figure out the color of her hat. **She initially makes the assumption that she is wearing a green hat.**

i) If Lucy was wearing the green hat, would anybody have immediately known which color hat they had on?

Kaley: \_\_\_\_\_ "I don't know"

Emmanuelle: \_\_\_\_\_ "I don't know"

Lucy: \_\_\_\_\_ "I don't know"

ii) Does this agree with what you wrote for part a? \_\_\_\_\_ **Yes**

In the first round, Lucy knows the other two are Y and Y. Lucy herself does not know if her hat is G or Y. But if Lucy assumes that her own hat is G, then she would expect that neither K nor E would be able to guess either. After all, if Lucy had G, K would see G\_Y, and thus K would not know if the right answer was GYY or GGY, and meanwhile E would see GY\_ and now know if the right answer was GYY or GYG. This agrees with 5a: all three would say "I don't know" if Lucy had a green hat.

iii) After hearing everybody's answer, would anybody be able to figure out the color of their hat? (Remember, Lucy is assuming that she has a green hat on.)

Kaley: \_\_\_\_\_ **Yellow**

Emmanuelle: Yellow

Lucy: "I don't know"

iv) Does this agree with what you wrote for part b? No

v) If the answer is not the same as part b, explain what this contradiction means.

The second round, however, is more informative. If Lucy assumes she has a G, and she hears everyone say "I don't know" in the first round, then she will want to listen closely to the next round. After all, when it is K's turn to speak, K would be seeing "G\_Y" and K would know that L did not see "GG\_" because L did not yell out "Yellow!". So K would know that L saw "GY\_" last round, and thus K would right now yell, "Yellow!". But if K is silent, then K must be seeing "Y\_\_". That would explain why L was quiet last round: L saw "YY\_" and so did not guess. K's silence thus tells Lucy that Lucy must not have G.

vi) Does Lucy know which color hat she has on?

Yes, in the third round, Lucy can now confidently say she has Y. And, by the same process, everyone else also knows they have Y also

vii) Does everybody else know?

Yes, by the same process, everyone else also knows they have Y also