ORMC: DIOPHANTINE EQUATIONS

ORMC, OLYMPIAD GROUP 1, WEEKS 3–4

Problem 1. How many positive integers $n \leqslant 30,000$ satisfy $2 \mid n$, $3 \mid n-1$ and $5 \mid n-2$?

Problem 2.

(a) Find all pairs of integers (a, b) such that $3a + 5b = 1$.

(b) For what integers c does the equation $3a + 5b = c$ have a solution in integers a, b?

(c) Find all pairs of integers (a, b) such that $3a + 5b = 4$.

Problem 3.

(a) Find all pairs of integers (a, b) such that $a^2 + 5b^2 = 2^{2020}$.

(b) Find all pairs of integers (a, b) such that $a^3 + 5b^3 = 7^{2020}$.

Problem 4. Find all solutions in positive integers to the equation $a^2 + b^4 + c^{10} = 3 \cdot 2^{2020}$.

Problem 5. Find all integer solutions to the equation $a^3 + 3b^3 + 9c^3 = 0$.

Problem 6. Find all pairs of integers (a, b) such that $4a^2 - b^3 = 1$.

Problem 7. Below, n denotes a positive integer.

(a) Find all perfect cubes of the form $2^n - 1$.

(b) Find all perfect cubes of the form $3^n - 1$.

Problem 8.

(a) Find all pairs of positive integers (a, b) such that $2^a + 1 = 3^b$.

(b) Find all pairs of positive integers (a, b) such that $2^a - 1 = 3^b$.

(c*) Find all solutions in positive integers to the equation $2^a + 3^b = 5^c$ with $a \geqslant 2$.

Problem 9. Find all solutions in nonnegative integers to the equation $2^{2^a} + 2^{2^b} = 2^{2^c}$.

Problem *10. Show that the equation $2^a - a^3 = b!$ has only finitely many solutions in positive integers a, b.

Problem *11. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is never an integer for $n \geqslant 2$.

Problem *12. Find all *prime* numbers of the form $n^8 + 2^{2^n+2}$, where n is a positive integer.

Homework 1

Problem 1. Show that there are infinitely many integer values n, such that $7 \mid 2^n - 1$ and $11 \mid 3^n - 4$.

Problem 2. Solve the equation $a^2 + 2b^2 = 11 \cdot 2^n$, where a, b, n are positive integers.

Problem 3. Show that there are infinitely many positive integers n such that the expression (i.e. polynomial) $x^n + n$ can be factored (i.e. written as a product of other non-constant polynomials with integer coefficients; for instance, $x^{27} + 27 = (x^9 + 3)(x^{18} - 3x^9 + 9)$).

Homework 2

Problem 4. Find all perfect squares of the form $5^n + 4$.

Problem 5. Solve the equation $7^a + 5 = 6^b$, where a and b are non-negative integers.

Problem 6. Show that the equation $2^a + 1 = 11^b$ does not have any solutions in positive integers a and b.