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Intro to geometry, or is your world flat?

Part 2

Definition 1 *A great circle on a 2D sphere is a circle having the same radius as the sphere.*

Arcs of great circles are geodesic lines (shortest paths connecting points) on a sphere. Great circles are intersections of the sphere with 2D planes passing through the sphere's center.

Example 1 *The circles containing meridians (a meridian is half a circle connecting the poles) and the Equator are great circles on the globe.*

Definition 2 *An angle between two great circles is the dihedral angle between the planes containing them.*

Example 2 *The angle between the great circle containing a meridian and the Equator is $\pi/2$.*

Definition 3 *Let O be the center of the sphere. Let r be its radius. Let A and B be two points on the sphere. We define the spherical distance between A and B as the product of the radius and the angle AOB measured in radians.*

$$||AB||_s = r \angle AOB$$

Problem 1 *The mean radius of the Earth is 6,371 km.¹ What is the spherical distance between the North and South poles? Between a pole and any point on the Equator?*

¹The Earth is not a perfect sphere.

The first accurate measurement of the Earth's radius was done by Eratosthenes². He noticed that at the geographic noon of the longest day of the year (midsummer or summer solstice), the Sun in his home city of Alexandria was $1/50$ ($7^\circ 12'$) of the full circle South of the zenith. He then traveled South, to the city of Syene, now Aswan in Egypt, by camel, counting the number of the camel's steps throughout the journey. In modern units, his trip was about 800 km long. During the summer solstice, the Sun was directly overhead in Syene. Using these data, the great Greek was able to compute the radius of the Earth.

Problem 2 *According to Eratosthenes, what was the radius of the Earth?*

Problem 3 *How would you measure the angle between the zenith and the position of the Sun in the sky?*

Definition 4 *Two points, P and P' , on a sphere are called antipodal, or antipodes, if they belong to a straight line passing through the center of the sphere.*

Recall that in the Euclidean plane, there exists a unique straight line passing through any pair of different points.

Problem 4 *Prove that for any pair of different points on a sphere that are not antipodes, there exists a unique great circle passing through them. How many great circles pass through a pair of antipodes?*

Definition 5 *Let l be a great circle on a sphere. Let us call its poles the points P_l and P'_l of the sphere lying on the straight line orthogonal to the plane of the circle and passing through the center of the sphere. Conversely, the polar of a point P on the sphere is the great circle lying in the plane orthogonal to the straight line PP' .*

For any straight line and a point in the Euclidean plane, there exists a unique straight line passing through the point and orthogonal to the original line.

²276 – 195 BC

Problem 5 Prove that for a great circle and a point that is not its pole, there exists a unique great circle passing through the point and orthogonal to the original circle. How many great circles orthogonal to a given one and passing through its pole are there on the sphere?

For any point P of the euclidean plane and any positive real number ρ , there exists a unique circle of radius ρ centered at P . Similarly, for any point P of a sphere and for any real number

$$0 < \rho \leq \pi r$$

there exists a unique circle on the sphere centered at P and of the (spherical) radius ρ .

Problem 6 What is the circle centered at P' and having the (spherical) radius $\pi r - \rho$?

Any two great circles intersect at two antipodal points. There are no parallel lines in spherical geometry. The lines divide the sphere into four *biangles*, spherical polygons having two sides and vertices. The main parameter of a biangle is the angle α between the Euclidean planes containing its sides.

Fact 1 The area S of a sphere is $4\pi r$.

Problem 7 What is the area $S_{\frac{\pi}{2}}$ of a biangle with the angle $\pi/2$?

Problem 8 What is the area S_{α} of a biangle with the angle α ?

Theorem 1 Let ABC be a spherical triangle with the angles α , β , and γ . Then the area S_{ABC} of the triangle is $r(\alpha + \beta + \gamma - \pi)$.

Problem 9 Prove the theorem.

For the angles α , β and γ measured in degrees,

$$\alpha + \beta + \gamma = 180^\circ \left(1 + \frac{S_{ABC}}{\pi r} \right).$$

Corollary 1 The sum of the angles of a spherical triangle is greater than 180° .

The following part of this lecture is about navigation, the art of pinpointing oneself on the surface of the Earth.

Question 1 *What do the A.M. and P.M. abbreviations mean?*

Problem 10 *It's 4:00 P.M. in Los Angeles, CA. By looking at the model globe, find the time in Moscow, Russia.*

Question 2 *What are the meanings of the GMT and UTC abbreviations and of the expression "Zulu time"?*

Definition 6 *A longitude is a geographic coordinate that specifies the East-West position of a point on the Earth's surface.*

Problem 11 *How to measure a longitude with a watch?*

Definition 7 *The latitude is the angle between the direction to the center of the Earth and the equatorial plane.*

As the night sky rotates around an observer located in the Northern hemisphere, the North Star remains still.

Problem 12 *Use the above fact to devise a way of finding one's latitude.*

Question 3 *What is the International Date Line and what do we need it for?*

If time permits, we shall also discuss some elliptic geometry, that of a sphere with pairs of antipodal points "glued together" into a single point.