GAUSSIAN INTEGERS II

OLGA RADKO MATH CIRCLE
ADVANCED 2
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This week, we will continue to investigate the irreducible elements of $\mathbb{Z}[i]$ and eventually characterize the integers which are sums of two squares. Last week, we showed that prime integers that are congruent to 3 mod 4 can not be written as sums of two squares and therefore are irreducible in $\mathbb{Z}[i]$. Now we have to analyze the more difficult case of when $p \equiv 1 \bmod 4$.

Problem 1.
(a) Find an integer a such that $a^4 \equiv 1 \bmod 5$ but $a^k \not\equiv 1 \bmod 5$ for any $0 < k \leq 3$.
(b) Find an integer a such that $a^6 \equiv 1 \bmod 7$ but $a^k \not\equiv 1 \bmod 7$ for any $0 < k \leq 5$.

It turns out that this is always possible. If p is any prime integer, then there exists some $0 \leq a \leq p-1$ such that $a^{p-1} \equiv 1 \bmod p$ but $a^k \not\equiv 1 \bmod p$ for any $0 \leq k \leq p-2$. Such an a is called a *primitive root* mod p.

Problem 2.
(a) Is 2 a primitive root mod 7?
(b) Is 2 a primitive root mod 11?
(c) Is 3 a primitive root mod 17?

Another fact: We know that if x is an integer such that $x^2 = 1$, then $x = 1$ or -1. This is also true mod p, i.e. if x is an integer such that $x^2 \equiv 1 \bmod p$, then $x \equiv 1$ or $-1 \bmod p$. Using these two facts, prove the following.

Problem 3. If $p \equiv 1 \bmod 4$, prove that there is some integer n such that p divides $n^2 + 1$. (Hint: This is equivalent to showing that some n satisfies $n^2 \equiv -1 \bmod p$. Let a be a primitive root mod p and proceed).

Now we are ready to analyze the case when $p \equiv 1 \bmod 4$.

Problem 4. The purpose of this exercise is to prove that if $p \equiv 1 \bmod 4$, then p factors as $p = (a + bi)(a - bi)$ where $a + bi$ is an irreducible element of $\mathbb{Z}[i]$.
(a) Factor $n^2 + 1$ in the Gaussian integers for any integer n.
(b) Let p be a prime integer congruent to 1 mod 4 and let n be any integer. Show that p does not divide $n + i$ via a contradiction argument. (Hint: What can we say about p and $n - i$?)
(c) By the claim above, p divides $n^2 + 1$ for some integer n. Prove that p is not irreducible.
(d) Show that p factors as $p = (a + bi)(a - bi)$ for integers a, b. (Hint: Problem 8(a) from last week.)
(e) Show that $a + bi$ and $a - bi$ are irreducible Gaussian integers. (Hint: Use the norm.)

We are now ready to write down all irreducible elements of $\mathbb{Z}[i]$. As a recap of what we have done, there are three classes of irreducible elements in the Gaussian integers.

1. We know that $1 + i$ is irreducible via the norm.
2. We showed that prime integers congruent to 3 mod 4 are irreducible.
3. Finally, we showed that when p is a prime integer congruent to 1 mod 4, the distinct irreducible factors $a + bi$ and $a - bi$ of $p = a^2 + b^2$ are irreducible.

We want to show that these are all the irreducible elements of the Gaussian integers.

Problem 5. Assume that $\alpha = a + bi$ is an irreducible element of $\mathbb{Z}[i]$.
(a) Prove that α divides $N(\alpha)$.
(b) Conclude that α divides some prime integer. (Hint: $N(\alpha)$ is an integer that might not be prime.)
(c) Conclude that α must be an element of our list.

Now, finally, we are able to prove a complete characterization of which positive integers are sums of two squares. The following theorem was first proved by Fermat.

Theorem 1. Let n be a positive integer. Write the prime factorization of n as

$$n = 2^k \cdot p_1^{e_1} \cdots p_\ell^{e_\ell} \cdot q_1^{f_1} \cdots q_d^{f_d}$$

where $p_1, \ldots, p_\ell$ are distinct primes congruent to 1 mod 4 and $q_1, \ldots, q_d$ are distinct primes congruent to 3 mod 4. Then n is the sum of two squares if and only if all of the f_j are even.

Problem 6. Prove the above theorem.

(a) Prove that n is the sum of two squares if and only if there is some Gaussian integer $\gamma = A + Bi$ such that $N(\gamma) = n$.

(b) Prove that if α is irreducible in $\mathbb{Z}[i]$, then $N(\alpha)$ is equal to 2, a prime congruent to 1 mod 4, or the square of a prime congruent to 3 mod 4.

(c) Suppose $n = N(\gamma)$ for some $\gamma \in \mathbb{Z}[i]$. Show that each f_j must be even (Hint: Factor $\gamma = \alpha_1 \cdots \alpha_m$ as a product of irreducible Gaussian integers. Take the norm and use part (b).)

(d) Suppose that each f_j is even. Show that there exist irreducible Gaussian integers $\alpha_1, \ldots, \alpha_m$ such that $N(\alpha_1) \cdots N(\alpha_m) = n$. (Hint: Problem 8(c) from last week.)

(e) Explain why parts (a)-(d) together complete the proof of the theorem.

Problem 7. (CHALLENGE). Prove that if p is a prime integer and $a \not\equiv 0 \bmod p$, then $a^{p-1} \equiv 1 \bmod p$. (Hint: Compare the two sets $\{1, 2, 3, \ldots, p-1\}$ and $\{a, 2a, 3a, \ldots, (p-1)a\}$.) This result is known as *Fermat's Little Theorem.*