Hat Problems and the Axiom of Choice

Aaron Anderson and Glenn Sun

October 25, 2020

1 Introduction

The Axiom of Choice is quite infamous for having many surprising consequences. Today we'll take a deeper look at its simple statement and how it leads to some surprises.

Motivation 1.1 (Hat Problem I)**.** Imagine an infinite line of people. Each person will be given a hat with a natural number on it. (The numbers may repeat. Each person will see the hats of everyone else but not their own. No communication is allowed after receiving hats.) After everyone is given their hat, everyone will be asked to simultaneously guess the number on their own hat. Under the best possible collective strategy, how many people can guess correctly? How many guess incorrectly?

Of course, physical constraints in our universe prevent infinite lines of people from existing. We also have to assume that people can perform infinite computations in their heads. But even with infinite computational power, it just seems impossible that by seeing only other peoples' hats, you can know your own.

Motivation 1.2 (Hat Problem II)**.** The setup is almost the exactly the same as the previous problem, but now people guess in order. When the first person guesses, everyone else can hear their guess, and so on. What's the best strategy now?

Motivation 1.3 (Banach-Tarski paradox)**.** It's possible to chop a sphere into 5 pieces, rotate them around without deforming the pieces, and put them back together to make two perfect spheres, each of the same volume as the original sphere (with no holes!).

---

2 Equivalence Relations

Before we get to the Axiom of Choice, we have to build up some vocabulary to allow us to understand what it says.

Definition 2.1 (equivalence relation)**.** A relation $\sim$ over a set S is something that says if any two elements of S are related ($a \sim b$) or unrelated ($a \not\sim b$) [1]. An equivalence relation is a special kind of relation that further satisfies three properties:

1. Reflexivity: for all $a \in S$, we have $a \sim a$.
2. Symmetry: for all $a, b \in S$, if $a \sim b$, then $b \sim a$.
3. Transitivity: for all $a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

One example of an equivalence relation is just equality. Take any set S. Then,

1. Reflexivity: for all $a \in S$, we indeed have $a = a$.
2. Symmetry: for all $a, b \in S$, if $a = b$, then indeed $b = a$.
3. Transitivity: for all $a, b, c \in S$, if $a = b$ and $b = c$, then indeed $a = c$.

Most equivalence relations aren't this straightforward, but still fairly straightforward. One useful way to think about equivalence relations is just "kind of equal".

Exercise 2.2. Which of the following are equivalence relations? Justify answers.

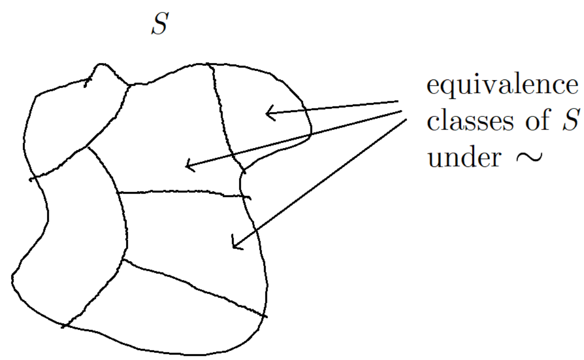
1. The relation $\equiv \pmod{n}$ over the integers $\mathbb{Z}$, where $a \equiv b \pmod{n}$ means a and b have the same remainders mod n.
2. The relation $\leq$ over the real numbers $\mathbb{R}$, where $a \leq b$ means a is less than or equal to b.
3. The relation $\sim$ over triangles, where $\triangle ABC \sim \triangle A'B'C'$ means all corresponding angles are equal (i.e. $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, and $m\angle C = m\angle C'$.)
4. The relation $\sim$ over pairs of integers, where $(a, b) \sim (c, d)$ means $ad = bc$.
5. The relation $\sim$ over pairs of real numbers, where $(a, b) \sim (c, d)$ means the distance from (a, b) to (c, d) is an integer (using the standard distance formula).

Definition 2.3 (equivalence class)**.** Given an equivalence relation $\sim$ on a set S, the equivalence class of a element $a \in S$ is the set of all $b \in S$ such that $a \sim b$.

Exercise 2.4. For every equivalence relation in Exercise 2.2, describe the equivalence classes, and determine how many equivalence classes there are.

A picture to have in mind is the following:

[1]More formally, a relation $\sim$ is a subset of S^2, the set of pairs of elements of S. We write $a \sim b$ as shorthand notation for $(a, b) \in \sim$, and $a \not\sim b$ as shorthand notation for $(a, b) \notin \sim$.



Informally, everyone inside an equivalence class is "kind of equal" to each other but "kind of not equal" to everyone else. A formal statement of the above picture is the following:

Definition 2.5 (partition)**.** A *partition* of a set S is a collection of nonempty subsets of S that are pairwise disjoint (no two different subsets share any elements) and whose union is S (together they make up all of S).

Proposition 2.6. The equivalence classes of an equivalence relation $\sim$ on a set S form a partition of S. Conversely, given any partition of S, one can construct an equivalence relation $\sim$ whose equivalence classes are exactly the subsets in the partition.

Exercise 2.7. Prove the above proposition.

3 The Axiom of Choice

The Axiom of Choice states the following:

Definition 3.1 (choice function, representative element)**.** Given an equivalence relation $\sim$ on a set S, a choice function is a function that chooses an element from every equivalence class. A chosen element is called the representative element of its class.

Axiom 3.2 (Choice)**.** Every equivalence relation has a choice function.

In other words, there is always a way to pick a representative element from every equivalence class (even if there are infinitely many equivalence classes).

Exercise 3.3. For each of the equivalence relations in Exercise 2.2, give an example of a choice function, explicitly specifying a representative element for each equivalence class.

In the previous exercise, you saw that for many equivalence relations, we can explicitly write down an example of a choice function. It's easier to come up with a choice function for some equivalence relations more than others. The Axiom of Choice says that a choice function exists, even if it's not obvious how to explicitly write it.

Now let's solve Hat Problem I. If you want to try completely by yourself, stop reading here. The following exercise outlines the crucial step.

Exercise 3.4. Define an equivalence relation $\sim$ on infinite sequences of natural numbers so that two sequences are equivalent if they are eventually the same.

1. Are the sequences $27, 0, 3, 1, 4, 1, 5, 9, \dots$ and $5, 2, 3, 1, 4, 1, 5, 9, \dots$ (digits of pi starting at the third position) equivalent?
2. Are the sequences $0, 1, 0, 1, \dots$ and $1, 0, 1, 0, \dots$ equivalent?
3. Verify that $\sim$ is an equivalence relation.

The above equivalence relation is yet another for which it might not be obvious how to write down a choice function. But the Axiom of Choice says that we can still pick one representative element from every equivalence class.

Exercise 3.5. Use the Axiom of Choice on the above equivalence relation, devise a strategy for Hat Problem I in which all but finitely many people guess correctly.

Before we get to Hat Problem II, here's an optional exercise that's unrelated to the Axiom of Choice, but has an important idea that Hat Problem II will use.

Exercise 3.6 (optional)**.** Give a way to pair every rational number with a natural number in a way such that no two rational numbers are paired to the same natural number. Explain how such a pairing allows you to unambiguously specify any rational number by saying a natural number.

The purpose of the above exercise is just to get the general idea that natural numbers can encode quite a lot more information than it may seem at first glance. A similar idea may help you in solving Hat Problem II (not necessarily this particular pairing).

Exercise 3.7. Explain the solution to Hat Problem II. (Hint below.[2])

Exercise 3.8. Consider a modification of Hat Problem II where each hat has a number from 0 to $n-1$ on it, for some natural number n. Hence, guesses are only allowed to be numbers from 0 to $n-1$. Can you find a similar strategy to solve this modification optimally?

4 The Right Inverse

You might think now that the Axiom of Choice is preposterous, since although it may look innocent at first, it leads to results that are completely unintuitive. The goal of this section

[2]Hint: Start with the idea as Hat Problem I. Notice that the sequence of numbers that the first person sees and the representative sequence of its equivalence class differ in only finitely many spots. Try to encode all the differences into a natural number!

is to show you that this isn't always the case — in some applications, the Axiom of Choice is *necessary* for intuitive statements to be proven.

For the first few exercises in this section, we won't be using the Axiom of Choice. We'll just be working with functions. Recall that a function f specifies an input set (domain) X, an output space (codomain) Y, and a rule [3] that maps each $x \in X$ to a unique output $y \in Y$. The output is often denoted by $f(x)$.

Definition 4.1 (injective, surjective, bijective)**.** A function $f : X \to Y$ is:

$$\left.\begin{array}{r}\text{injective}\\ \text{bijective}\\ \text{surjective}\end{array}\right\} \text{ if for all } y \in Y\text{, there exists } \left\{\begin{array}{c}\text{at most 1}\\ \text{exactly 1}\\ \text{at least 1}\end{array}\right\} x \in X \text{ such that } f(x) = y.$$

You may have heard these before as "one-to-one" (injective) and "onto" (surjective).

Exercise 4.2. Categorize the following functions as injective, surjective, bijective, or none of the above.

1. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.
2. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.
3. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x(x+1)(x-1)$.
4. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2^x$.
5. $f : \mathbb{R} \to (0, \infty)$ given by $f(x) = 2^x$.
6. $f : \mathbb{N}^2 \to \mathbb{N}$ given by $f(n, m) = 2^n 3^m$.

Definition 4.3 (left inverse, right inverse, inverse)**.** For a function $f : X \to Y$, a function $g : Y \to X$ is called a

$$\left.\begin{array}{r}\text{left inverse}\\ \text{inverse}\\ \text{right inverse}\end{array}\right\} \text{ if } \begin{cases} g(f(x)) = x \text{ for all } x \in X\\ \text{both}\\ f(g(y)) = y \text{ for all } y \in Y\end{cases}.$$

The following is an important basic property of bijective functions. Drawing a picture helps a lot with understanding why it is true.

[3] Formally, a function is a subset of $X \times Y$ that every $x \in X$ appears exactly once in the first component.

Proposition 4.4. A function $f : X \to Y$ is

$$\left.\begin{matrix}\text{injective}\\\text{bijective}\\\text{surjective}\end{matrix}\right\}\text{ if and only if it has a }\left\{\begin{matrix}\text{left inverse}\\\text{inverse}\\\text{right inverse}\end{matrix}\right. .$$

Here is the surprising part: the first two parts (injective $\iff$ left inverse, and bijective $\iff$ inverse) can be proven without using the Axiom of Choice. However, the third equivalence is not true unless you accept the Axiom of Choice!

Exercise 4.5. Prove the first two parts of the above proposition.

Exercise 4.6. Assume the Axiom of Choice. Prove the third part of the above proposition.

Exercise 4.7. Assume the negation of the Axiom of Choice. That is, assume that there exists an equivalence relation $\sim$ on a set S for which no choice function exists. Give an example of a surjective function with no right inverse.

5 Challenge: Banach-Tarski Lite

The Banach-Tarski paradox is possible because the axiom of choice lets you define sets that are so weird, it is impossible to calculate their volume. To see what this means, let's assume for a bit that every subset of $\mathbb{R}^3$ has a well-defined volume, and that volume satisfies a few obvious axioms. Then we will show from these assumptions that the Banach-Tarski paradox is impossible. This means that if you assume the axiom of choice, you can't have a well-defined volume for every subset of $\mathbb{R}^3$.

Axiom 5.1 (Axioms of Volume)**.** Let V be a function that assigns to (some) subsets of $\mathbb{R}^3$ a volume which is either a nonnegative real number or ∞. Then for V to really represent volume, the following should be true:

- The empty set has volume 0: $V(\emptyset) = 0$.
- If $A_1, A_2, A_3, \ldots$ are pairwise disjoint subsets of $\mathbb{R}^3$ with $V(A_i)$ defined for each i, then $V(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} V(A_i)$. (Any sum of ∞ plus anything should also be defined to be ∞.)
- The area of the unit sphere is finite and positive.
- If $A \subseteq \mathbb{R}^3$ has a defined volume, and B is the set formed by moving or rotating A, then $V(A) = V(B)$.

Exercise 5.2. Show that if $A \subseteq B \subseteq \mathbb{R}^3$, and A, B both have well-defined volume, then $V(A) \leq V(B)$. (We define $x \leq \infty$ for all $x \in \mathbb{R}$.)

Exercise 5.3. The Banach-Tarski paradox says that there exist five pairwise disjoint sets, A_1, A_2, A_3, A_4, A_5, such that $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ is the unit sphere, but you can move and/or rotate each A_i to a different set B_i, such that $B_1, \ldots, B_5$ are also pairwise disjoint, and $B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$ consists of two disjoint copies of the unit sphere.

Show that if $V(A_i)$ is defined for each i, then this is impossible.

Now we will show how the Axiom of Choice implies an easier version of the Banach-Tarski paradox, in a 1-dimensional context. To do this, we will use a "volume" function that satisfies basically the same axioms as before, except instead of applying to subsets of $\mathbb{R}^3$, it applies to subsets of $\mathbb{R}$. We want all the same axioms, except instead of assuming that the unit sphere has positive, finite volume, we want to assume that for every $a \leq b$, $V([a, b]) = b - a$, the length of the interval $[a, b]$, so V really measures some generalized version of "length" of a set. We will show directly that there is a set X such that $V(X)$ cannot be defined, much like the 5 subsets of the sphere in the Banach-Tarski paradox. The procedure will also involve cutting up a piece of (this time 1-dimensional) space, this time into countably many pieces, moving them, and reassembling them into a bigger region of space.

Exercise 5.4. For $x, y \in [0, 1]$, define $x \sim y$ to be true if and only if $y - x \in \mathbb{Q}$.

Show that $\sim$ is an equivalence relation.

Exercise 5.5. Using the axiom of choice, define $X \subseteq [0, 1]$ such that X contains exactly one representative of each equivalence class of $\sim$.

Exercise 5.6. We set up a contradiction:

- Show that there is a bijection $f : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$.
- For $k \in \mathbb{N}$, let X_k be $\{x + f(k) : x \in X\}$. Show that $[0, 1] \subseteq \bigcup_{k \in \mathbb{N}} X_k \subseteq [-1, 2]$.
- Show that $X_1, X_2, X_3, \ldots$ are pairwise disjoint.

Exercise 5.7. In each of the following cases, compute $V(\bigcup_{x \in \mathbb{N}} X_k)$ and derive a contradiction.

- $V(X) = 0$
- $V(X)$ is finite and positive
- $V(X) = \infty$

Conclude that $V(X)$ cannot be defined.

Our set X is called a *Vitali* set, and is the easiest example of a non-measurable set to construct and prove from scratch, using of course, the vital ingredient, the Axiom of Choice. This was discovered before the Banach-Tarski paradox, and Banach and Tarski were inspired by it. Unfortunately (or perhaps fortunately, if you don't like paradoxes), the Banach-Tarski result itself, of cutting up an obviously measurable set into finitely many pieces and reassembling into multiple copies of the original, or a larger copy of the original, doesn't work very well in 1 or 2 dimensions. The actual Banach-Tarski proof revolves around studying the group theory of 3-dimensional rotations and translations, to set up a very particular equivalence relation which we can then apply the axiom of choice to.