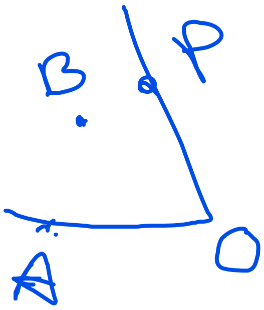
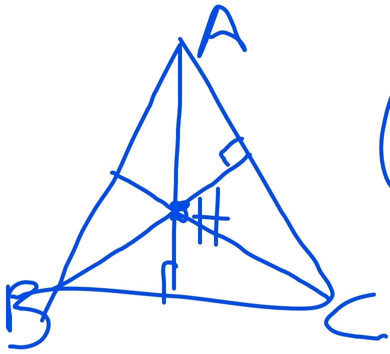
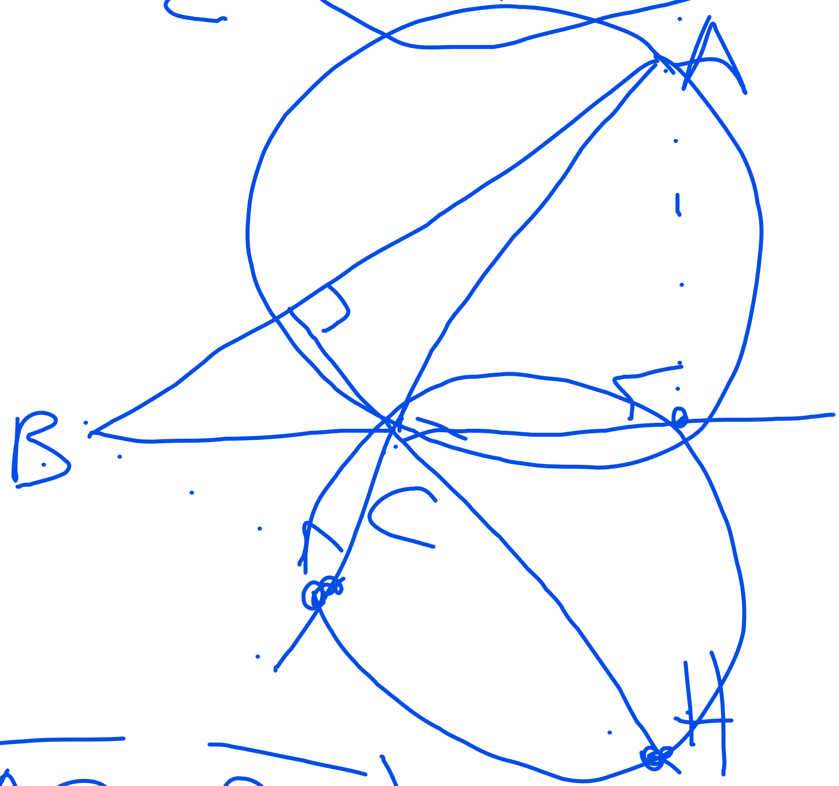
Directed Angles

What is the issue?

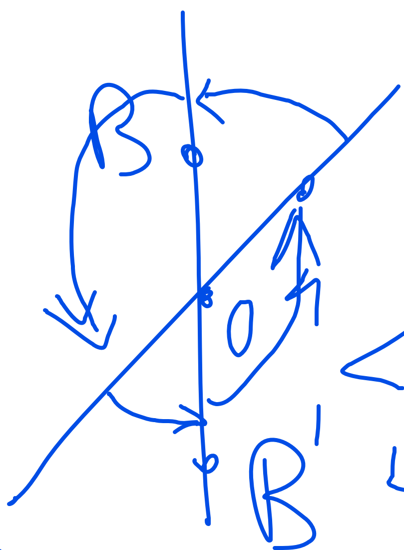




$$\measuredangle AOB + \measuredangle BOP = \measuredangle AOP$$



How to fix it?



$$\underbrace{\angle(\overline{AO}, \overline{BO})}_{+} = \alpha$$

$$0 = 0/2 \pmod{180} \quad \angle(\overline{BO}, \overline{AO})$$

$$= 180/2 = 90$$

Oblivion. $\measuredangle APA = 0$.

Anti-Reflexivity. $\measuredangle ABC = -\measuredangle CBA$.

Replacement. $\measuredangle PBA = \measuredangle PBC$ *if and only if* A, B, C *are collinear. (What happens when* $P = A$*?) Equivalently, if* C *lies on line* BA*, then the* A *in* $\measuredangle PBA$ *may be replaced by* C. ℓ m

Right Angles. *If* $\overline{AP} \perp \overline{BP}$*, then* $\measuredangle APB = \measuredangle BPA = 90^\circ$.

Directed Angle Addition. $\measuredangle APB + \measuredangle BPC = \measuredangle APC$.

$$\angle APA = \angle(\overline{AP}, \overline{AP}) = 0$$

$$\angle(\underset{\ell}{\overline{AB}}, \underset{m}{\overline{BC}}) + \angle(\underset{m}{\overline{CB}}, \underset{\ell}{\overline{BA}}) = 0$$

"WLOG this is diagram"

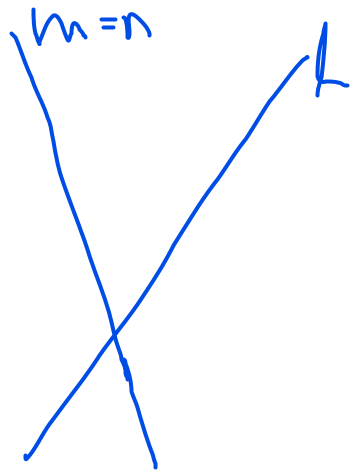
$\boxed{\text{* Package * ___ *}}$

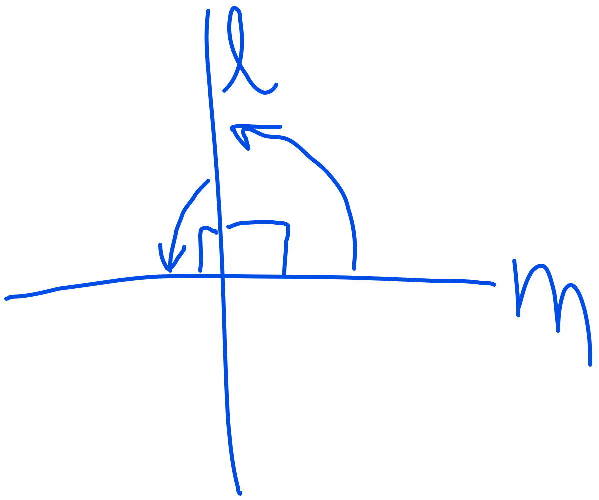
$\measuredangle PBA = \measuredangle PBC \Rightarrow A, B, C$ are collinear

$l = PB,\ m = BA,\ n = BC$

$\measuredangle(l, m) = \measuredangle(l, n) \Rightarrow A, P$

$\boxed{m = n}$





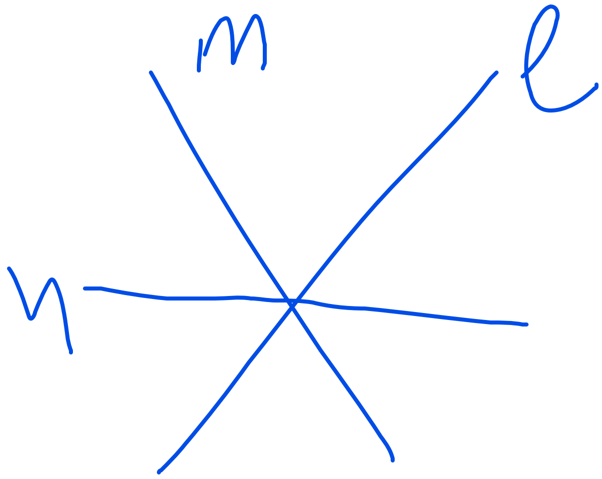
Theorem 3.4 (Triangles Sum to 180°)
For any lines ℓ, m, n we have

$$\measuredangle(\ell, m) + \measuredangle(m, n) + \measuredangle(n, \ell) = 0.$$

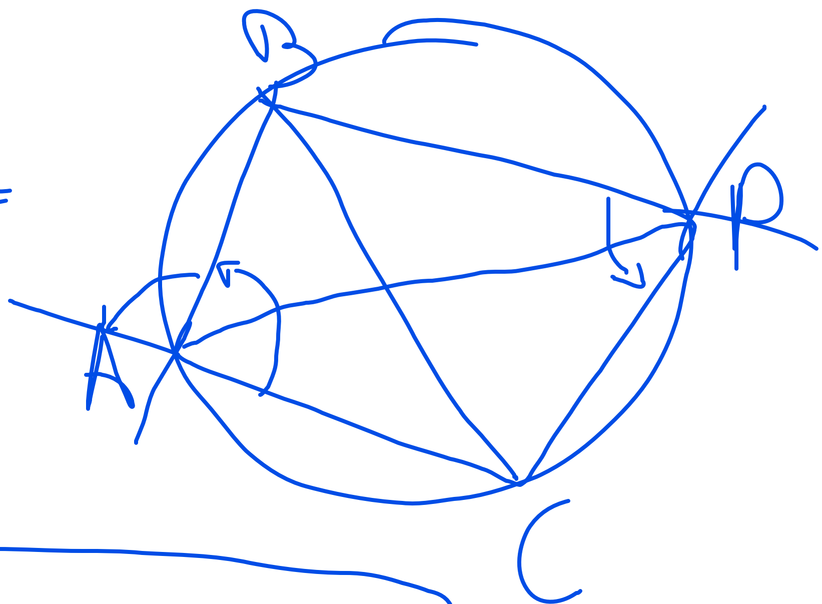
In particular, for any points A, B, C we have

$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0.$$

$\measuredangle(\ell, n) = -\measuredangle(n, \ell)$



Cyclic



$$\boxed{\measuredangle BAC = \measuredangle BPC}$$

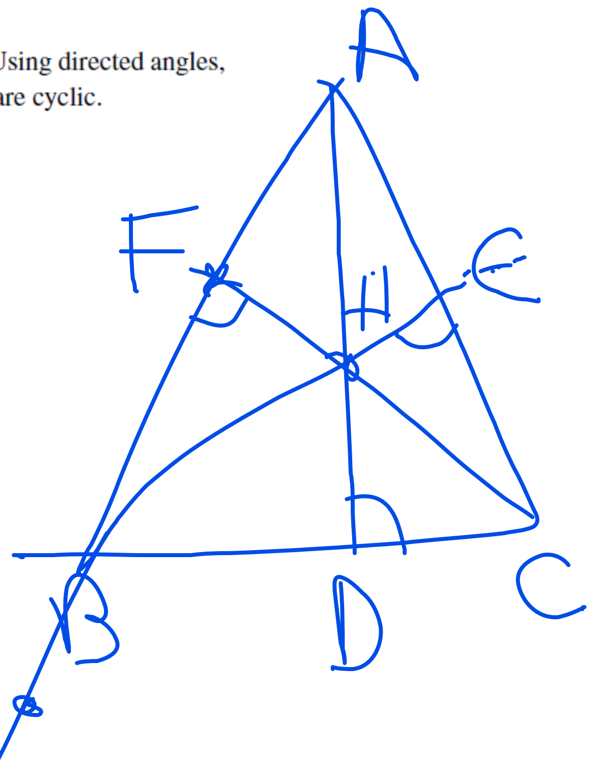
$\neq$ $\backslash\backslash$

$-\measuredangle CAB$ $\cancel{A}$ $90 - 0$

$\measuredangle CAB$

$2(90) = 2(0)$

Example 1.26. Let H be the orthocenter of $\triangle ABC$, acute or not. Using directed angles, show that $AEHF$, $BFHD$, $CDHE$, $BEFC$, $CFDA$, and $ADEB$ are cyclic.



E foot of perp. B on AC

D —||— A on BC

F —||— C on AB

$\angle HFB = \angle HDC$

$\measuredangle HFA = 90$ | $\measuredangle HEA = 90$

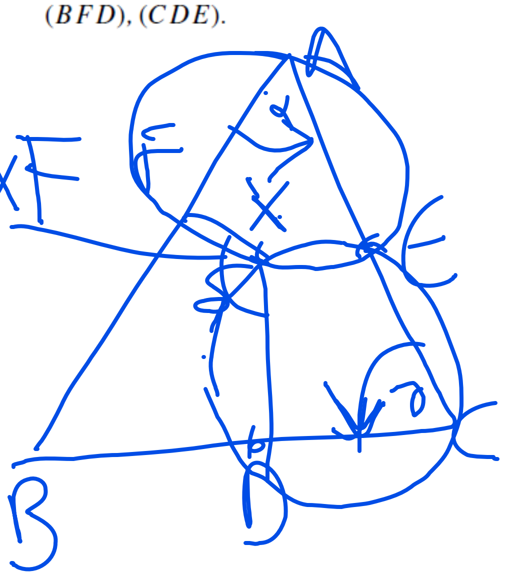
$\measuredangle(HF, FA)$

$\measuredangle(HF, AB) = 90$

$90 = -90$

Lemma 1.27 (Miquel Point of a Triangle). *Points D, E, F lie on lines BC, CA, and AB of $\triangle ABC$, respectively. Then there exists a point lying on all three circles (AEF), (BFD), (CDE).*

$\sphericalangle DXF$

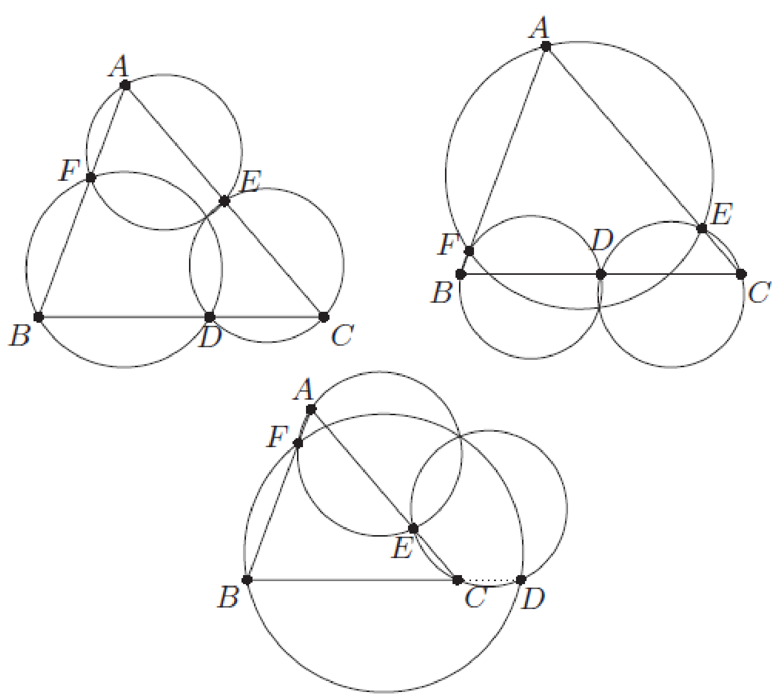


$\alpha + \gamma + \sphericalangle DXF$

$\sphericalangle FXE + \sphericalangle EXD + \sphericalangle DXF = 0$

$\alpha + \gamma + \sphericalangle CBA = 0$

$\alpha + \gamma + \sphericalangle DBF$



Lemma 1.18 (The Incenter/Excenter Lemma). *Let ABC be a triangle with incenter I. Ray AI meets (ABC) again at L. Let I_A be the reflection of I over L. Then,*

(a) *The points I, B, C, and I_A lie on a circle with diameter $\overline{II_A}$ and center L. In particular, $LI = LB = LC = LI_A$.*
(b) *Rays BI_A and CI_A bisect the exterior angles of $\triangle ABC$.*

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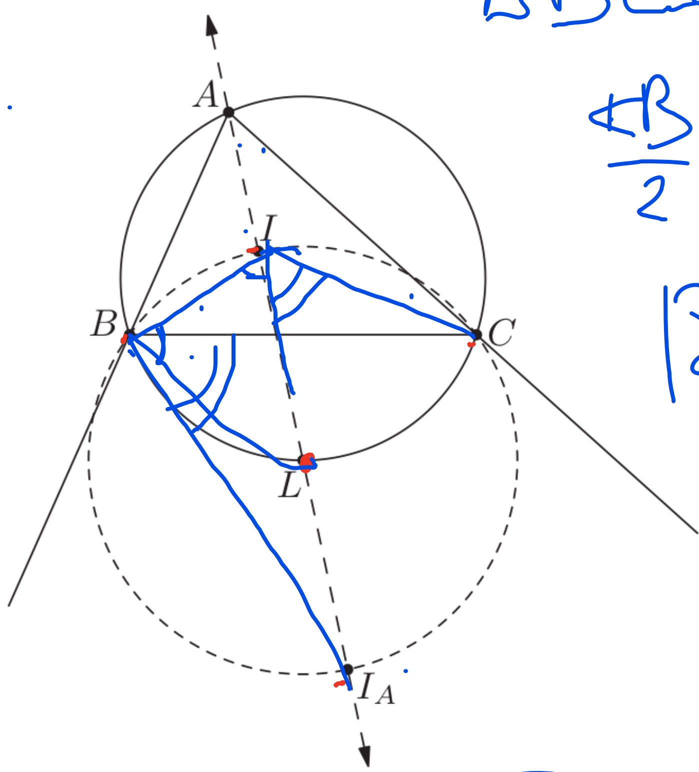
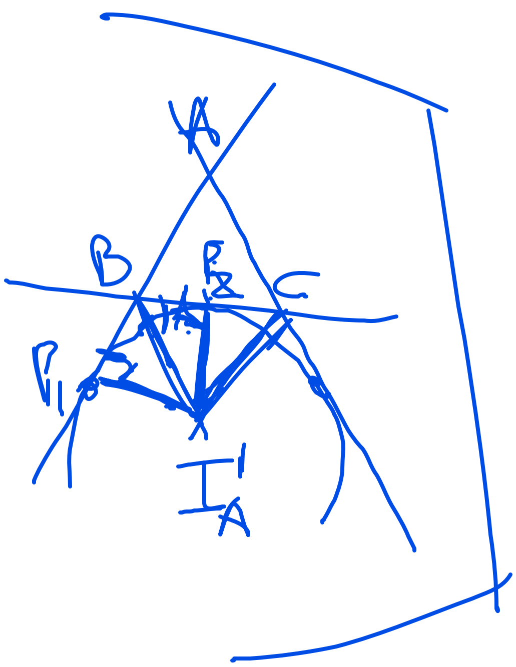
$\triangle BLI$

$\frac{\angle B}{2} + \frac{\angle A}{2}$

$180 - \angle B$





$\angle CBI_A = \angle CII_A$

$= \frac{\angle A + \angle C}{2}$

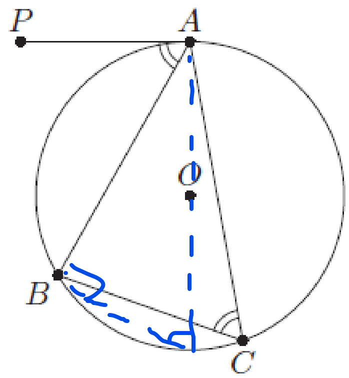
$= 90 - \frac{\angle B}{2}$

Proposition 1.31 (Tangent Criterion). *Suppose $\triangle ABC$ is inscribed in a circle with center O. Let P be a point in the plane. Then the following are equivalent:*

(i) $\overline{PA}$ is tangent to (ABC).
(ii) $\overline{OA} \perp \overline{AP}$.
(iii) $\measuredangle PAB = \measuredangle ACB$.

HW

(ii) $\Rightarrow$ (iii) $\quad \gamma + \beta = 90$



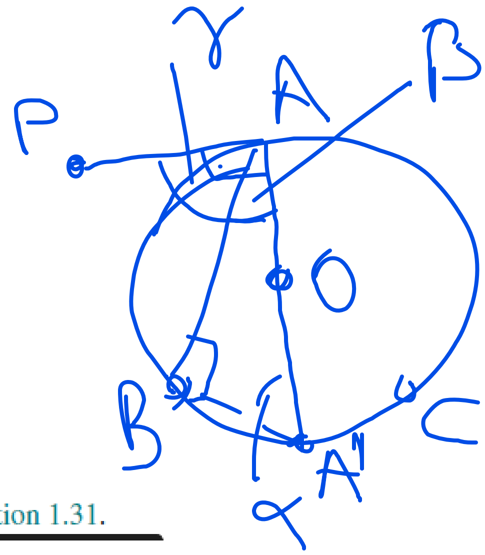
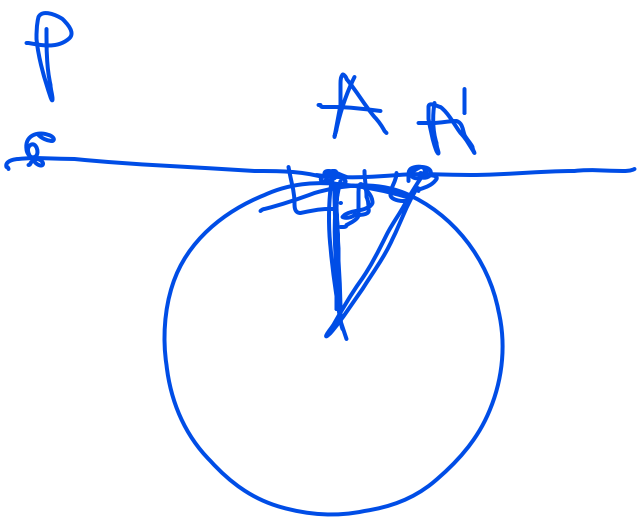
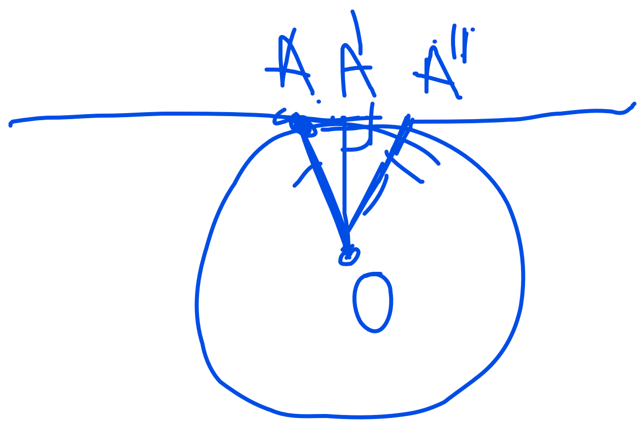


Figure 1.6A. PA is a tangent to (ABC). See Proposition 1.31.

$\beta + \alpha = 90$





Lemma 1.48 (Simson Line). *Let ABC be a triangle and P be any point on (ABC). Let X, Y, Z be the feet of the perpendiculars from P onto lines BC, CA, and AB. Prove that points X, Y, Z are collinear.* **Hints**: 278 502 **Sol:** p.243

$\measuredangle PYZ$
$= \measuredangle PAZ$
$= \measuredangle PAB$
$= \measuredangle PCB$
$= \measuredangle PCX$
$= \measuredangle PYX$

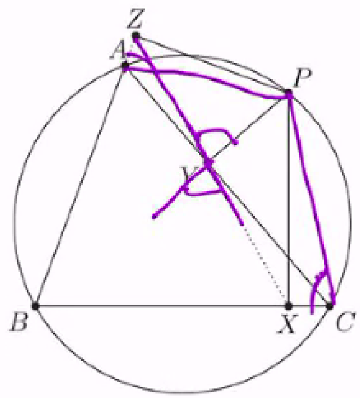
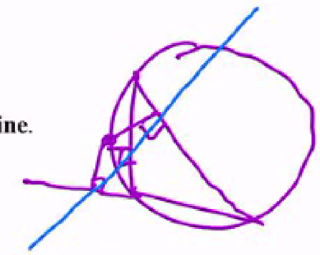


Figure 1.8D. Lemma 1.48; the Simson line.

$\measuredangle APC = \measuredangle ABC$

PZAY cyclic
PYXC cyclic
PZBX cyclic



$\measuredangle PYZ = \measuredangle PYX$

$\iff$ Y, Z, X collinear.