The following is known as a discriminant of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.

$$D = b^2 - 4ac$$  

(1)

**Theorem 1** If $D < 0$, then the quadratic equation $ax^2 + bx + c = 0$ with real coefficients $a \neq 0$, $b$, and $c$ has no real roots. If $D \geq 0$, then the following is the formula for the roots of the equation.

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$

(2)

**Problem 1** Prove Theorem 1.
Problem 2  
Find all the real solutions of the equation
\[ \sqrt{x - 2} = x - 4. \]

Problem 3  
Find all the real solutions of the equation
\[ 7 \left( x + \frac{1}{x} \right) - 2 \left( x^2 + \frac{1}{x^2} \right) = 9. \]
Problem 4 Sketch the graph of the function $y = ax^2 + bx + c$, given the following information: $a > 0, b > 0, D < 0$.

Is the coefficient $c$ positive, negative, or zero? Why?
Problem 5  Find all the real solutions of the equation

\[2x^2 + 6 - 2\sqrt{2}x^2 - 3x + 2 = 3x + 3.\]

Problem 6  Find all the real solutions of the equation

\[\sqrt[3]{x + a} + \sqrt[3]{x + a + 1} + \sqrt[3]{x + a + 2} = 0.\]
Vieta Formulas

**Theorem 2** Let $x_1$ and $x_2$ be the roots of the quadratic equation $ax^2 + bx + c$, $a \neq 0$. Then $x_1 + x_2 = -b/a$ and $x_1x_2 = c/a$.

**Problem 7** Prove Theorem 2.

**Problem 8** Write down a quadratic equation that has the roots $x_1 = 3$ and $x_2 = -4$. 


Problem 9  Generalize Vieta formulas to a cubic equation $ax^3 + bx^2 + cx + d$, $a \neq 0$.

Problem 10  Write down a cubic equation that has the roots $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. 
Problem 11 Without solving the equation $ax^2 + bx + c = 0$, find the sum of the squares of its roots provided that $a \neq 0$ and $D \geq 0$.

Problem 12 Find all the prime numbers $p$ and $q$ such that the equation $x^2 - px - q = 0$ has a solution that is a prime number.
A function $f(x)$ is called convex if for any $x_1$ and $x_2$ in its domain and for any $0 < \alpha < 1$,

$$f(\alpha x_1 + (1 - \alpha) x_2) < \alpha f(x_1) + (1 - \alpha) f(x_2).$$  \hfill (3)

**Problem 13** Give a geometric interpretation to formula (3).

**Problem 14** Prove that for a linear function $f(x) = bx + c$,

$$f(\alpha x_1 + (1 - \alpha) x_2) = \alpha f(x_1) + (1 - \alpha) f(x_2)$$

for any value of the parameter $\alpha$. 
Problem 15 Prove that \( f(x) = ax^2 + bx + c \) is convex for \( a > 0 \).

The value \( \hat{x} \) is called a minimum of a function \( f(x) \) if \( f(\hat{x}) \leq f(x) \) for every \( x \) in the function’s domain.

Problem 16 Sketch the graph of a function having two minima.
Problem 17 The function $f(x)$ is convex. Prove that it can have at most one minimum.

Problem 18 Find the minimum of the function $f(x) = ax^2 + bx + c$, $a > 0$. Prove that it is indeed a minimum. What is the value of the function at the point?
Problem 19  
Find the minimum of the function 
\[ f(x) = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2. \]

Problem 20  
A straight line in the plain is given by the equation \[ ax + by + c = 0. \] Find the distance from the point \((x_0, y_0)\) to the line.
Problem 21  Prove that for $x > 0$, $x + \frac{1}{x} \geq 2$.

Problem 22  Given $x + y + z = 1$, $x > 0$, $y > 0$, and $z > 0$, prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9.$$