COMPETITION-STYLE WARM UP

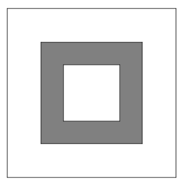
OLGA RADKO MATH CIRCLE
ADVANCED 2
OCTOBER 11, 2020

Problem 1. Can you draw a path on the surface of Rubik's cube (3x3x3 cube) that goes through every single square on the surface? The path should not go through any vertices.

The answer is yes. They can draw an example on a net of the cube.

Problem 2 (2010 AIME II Problem 2 ©MAA)**.**
A point P is chosen at random in the interior of a unit square S. Let $d(P)$ denote the distance from P to the closest side of S. Find the probability that $1/5 \leq d(P) \leq 1/3$.

Solution: The event in question corresponds to the shaded region in the following picture.



The innermost square has vertices $(1/3, 1/3), (1/3, 2/3), (2/3, 2/3)$, and $(2/3, 1/3)$. The middle square has vertices $(1/5, 1/5), (1/5, 4/5), (4/5, 4/5)$, and $(4/5, 1/5)$. The outer square has vertices $(0, 0), (0, 1), (1, 1)$, and $(1, 0)$. Since the point P is picked uniformly, the probability in question is just the area of the shaded region, which is

$$\left(\frac{3}{5}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{56}{225}.$$

Problem 3 (2002 AMC 12A Problem 16 ©MAA)**.**
Tina randomly selects two distinct numbers from $\{1, 2, 3, 4, 5\}$. Sergio randomly selects one number from the set $\{1, 2, \ldots, 10\}$. What is the probability that Sergio's number is greater than the sum of the two numbers chosen by Tina?

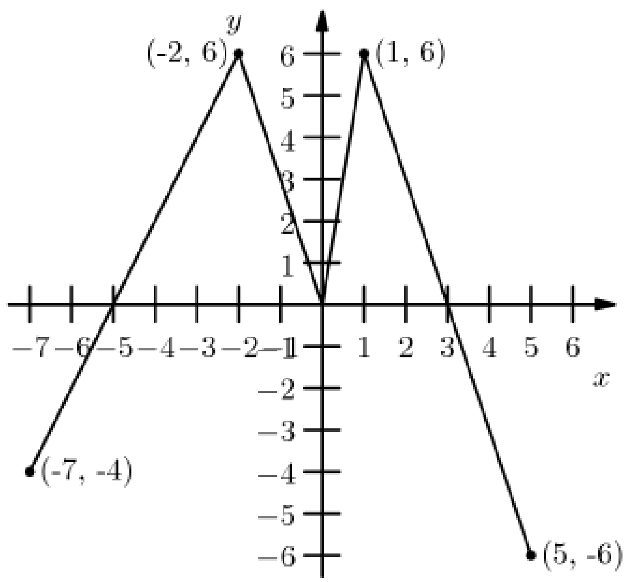
Solution: Let Sergio's number be S and Tina's numbers be T_1 and T_2. First note there are $\binom{5}{2} = 10$ choices for the unordered pair $\{T_1, T_2\}$. If $S = 1$ or $S = 2$ or $S = 3$, then $S > T_1 + T_2$ is impossible (because T_1 and T_2 are distinct). If $S = 4$, there is only one acceptable pair – $\{1, 2\}$ – so the probability of success is $1/10$ (success being defined as $S > T_1 + T_2$). Continuing in this way, we have

S	# of acceptable pairs	Probability of success
5	2	2/10
6	4	4/10
7	6	6/10
8	8	8/10
9	9	9/10
10	10	1

Then the total probability of success is obtained by averaging all of these –

$$\text{probability of success} \;=\; \frac{1}{10}\left(0+0+0+\frac{1}{10}+\frac{2}{10}+\frac{4}{10}+\frac{6}{10}+\frac{8}{10}+\frac{9}{10}+1\right) \;=\; \frac{2}{5}.$$

Problem 4 (2002 AMC 12A ©MAA)**.**
If $f : [-7, 5] \to \mathbb{R}$ is the function whose graph is shown below, how many solutions does the equation $f(f(x)) = 6$ have?



Solution: Letting $y = f(x)$, we see that $f(f(x)) = 6$ if and only if $f(y) = 6$, which according to the graph occurs exactly when $y \in \{-2, 1\}$. Hence $f(f(x)) = 6$ if and only if $f(x) \in \{-2, 1\}$. Drawing horizontal lines on the graph at $y = -2$ and $y = 1$, we see that $f(x) = -2$ for two values of x and $f(x) = 1$ for four values of x. Naturally since $-2 \neq 1$, these x-values must be distinct from each other, so in total we have 6 solutions to $f(f(x)) = 6$.

Problem 5 (1983 AIME ©MAA)**.**
Find the product of all real solutions to $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$.

Solution: Let $u = \sqrt{x^2 + 18x + 45}$, then our equation becomes $u^2 - 15 = 2u$, giving us $u = 5, -3$. Now u must be non-negative, so we have $u = 5$, giving us $x^2 + 18x + 45 = 25$, or $x^2 + 18x + 20 = 0$. Then it follows that the product of the roots must be 20, and so the last thing to check is that the roots are real. Since $18^2 - 80 > 0$, the roots are real and so the answer is 20.

Problem 6. Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.

Solution: (AoPS) Consider a 4×82 rectangle of points in the plane, such as $\{(x, y) \in \mathbb{Z}^2 : 0 \le x \le 3, 0 \le y \le 81\}$. For each column, there are 4 points and 3 possible colors per point, for a total of $3^4 = 81$ possible colorings. With 82 columns, by the Pigeonhole Principle, there are two columns with the same coloring. Also, there are 4 points per column and 3 possible colors, so by the Pigeonhole Principle, some color appears twice. From each of the two columns, take some corresponding two points of a color that appears twice. These form a rectangle all of whose vertices are the same color.

Problem 7 (2006 AIME I Problem 3 ©MAA)**.**
Find the least positive integer such that when its left-most digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

Solution: Suppose the original number is $N = \overline{a_n a_{n-1} \ldots a_1 a_0}$, where the a_i are digits and the first digit, a_n, is nonzero. Then the number we create is $N_0 = \overline{a_{n-1} \ldots a_1 a_0}$, so

$$N = 29N_0.$$

But N is N_0 with the digit a_n added to the left, so $N = N_0 + a_n \cdot 10^n$. Thus,

$$N_0 + a_n \cdot 10^n = 29N_0$$
$$a_n \cdot 10^n = 28N_0.$$

The right-hand side of this equation is divisible by seven, so the left-hand side must also be divisible by seven. The number 10^n is never divisible by 7, so a_n must be divisible by 7. But a_n is a nonzero digit, so the only possibility is $a_n = 7$. This gives

$$7 \cdot 10^n = 28N_0$$

or

$$10^n = 4N_0.$$

Now, we want to minimize both n and N_0, so we take $N_0 = 25$ and $n = 2$. Then

$$N = 7 \cdot 10^2 + 25 = 725,$$

and indeed, $725 = 29 \cdot 25$.

Similarly, you can see that $29a_0 \equiv a_0 \bmod 10$, and so $a_0 = 0, 5$. Then just try some numbers.

Problem 8 (2004 Manhattan Mathematical Olympiad)**.**
Seven line segments, with lengths no greater than 10 inches, and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle.
(**Hint:** Order the line segments by increasing length. Given the first two segments, what size would the third have to be to not form a triangle? Continue this way through the list of segments.)

Solution: Let the seven line segments a, b, c, d, e, f, g be ordered by increasing length. Suppose that no triangle can be formed by any three of them. Then it follows that $c \ge a + b$, and since $a, b \ge 1$, we have $c \ge 2$. Similarly, $d \ge b + c \ge 1 + 2 = 3$. Following this pattern, we have that $e \ge 5, f \ge 8, g \ge 13$. But this is a contradiction since all lengths must be at most 10. Therefore, one can choose three of them to represent the sides of a triangle.

Problem 9 (1988 AIME ©MAA)**.**
Suppose there is a function f defined on the set of ordered pairs (x, y) of positive integers which satisfies

$$f(x,x) = x, \tag{1}$$
$$f(x,y) = f(y,x), \quad \text{and} \tag{2}$$
$$(x+y)f(x,y) = yf(x,x+y). \tag{3}$$

Show that there is only one possible value of $f(14, 52)$ and find it.

Solution: The point here is that we can use the given identities to express $f(14, 52)$ in terms of something of the form $f(x, x)$, whose value we are given is x. Since our goal is to make the arguments of the function equal, it makes sense to try to do so by making them both small; after all, there are only n positive integers less than or equal to n, so our chances of finding overlap are good for small n. To do so, first note that the third given equation allows us to replace one argument by the difference between the two (provided it remains positive) since

$$(x+y-x)f(x,y-x) = (y-x)f(x,y-x+x) \implies f(x,y) = \frac{y}{y-x}f(x,y-x). \tag{4}$$

Using this, together with the second given equation,

$$\begin{aligned} f(14,52) &= \frac{52}{38}f(14,38) = \frac{52}{38}\cdot\frac{38}{24}f(14,24) = \frac{52}{38}\cdot\frac{38}{24}\cdot\frac{24}{10}f(14,10) \\ &= \frac{52}{10}f(10,14) = \frac{52}{10}\cdot\frac{14}{4}f(10,4) \\ &= \frac{13\cdot 7}{5}f(4,10) = \frac{13\cdot 7}{5}\cdot\frac{10}{6}f(4,6) = \frac{13\cdot 7}{5}\cdot\frac{10}{6}\cdot\frac{6}{2}f(4,2) \\ &= 13\cdot 7\cdot f(2,4) = 13\cdot 7\cdot\frac{4}{2}f(2,2) \\ &= 13\cdot 7\cdot 2\cdot 2 = 364. \end{aligned}$$

We remark that there is such a function, namely $f(x, y) = \text{lcm}(x, y)$, and indeed, 364 is the least common multiple of 14 and 52.

Problem 10 (2008 AIME I Problem 11 ©MAA)**.**
Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?
(**Hint:** Let a_n and b_n denote, respectively, the number of sequences of length n ending in A and B. Can you relate a_n and b_n to $a_{n-1}, a_{n-2}, b_{n-1}, b_{n-2}$?)

Solution: (AoPS) Let a_n and b_n denote, respectively, the number of sequences of length n ending in A and B. If a sequence ends in an A, then it must have been formed by appending two As to the end of a string of length $n-2$. If a sequence ends in a B, it must have either been formed by appending one B to a string of length $n-1$ ending in an A, or by appending two Bs to a string of length $n-2$ ending in a B. Thus, we have the recursions

$$a_n = a_{n-2} + b_{n-2}$$
$$b_n = a_{n-1} + b_{n-2}$$

By counting, we find that $a_1 = 0, b_1 = 1, a_2 = 1, b_2 = 0$.

n	a_n	b_n	n	a_n	b_n
1	0	1	8	6	10
2	1	0	9	11	11
3	1	2	10	16	21
4	1	1	11	22	27
5	3	3	12	37	43
6	2	4	13	49	64
7	6	5	14	80	92

Therefore, the number of such strings of length 14 is $a_{14} + b_{14} = \boxed{172}$.

Note: For the next problem, it may be helpful to recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and that if $z = e^{\frac{2\pi i}{n}}$, then $1 + z + z^2 + ... + z^{n-1} = 0$.

Problem 11 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.** Show that

$$\cos\frac{2\pi}{2n+1} + \cos\frac{4\pi}{2n+1} + \cdots + \cos\frac{2n\pi}{2n+1} = -\frac{1}{2}.$$

(**Hint:** Think about where each term in the sum sits on the circle geometrically and use the equations in the note above the problem.)

Let $z = e^{\frac{2\pi i}{2n+1}}$. Then we have that the real part of z^k is equal to the real part of z^{-k}, since they are complex conjugates of each other. Now we use the fact that $2n+1$ is odd, which allows us to pair each z^k with $z^{2n+1-k} = z^{-k}$. Then we have that the real part of $1+z+ ...+z^{2n}$ is equal to the real part of each term in the sum, and after pairing up all of the z^k, we have $1+2(\text{Re}(z)+\text{Re}(z^2)+...+\text{Re}(z^n)) = 1+2(\cos\frac{2\pi}{2n+1}+\cos\frac{4\pi}{2n+1}+\cdots+\cos\frac{2n\pi}{2n+1})$. Since $1 + z + ... + z^{2n} = 0$, we have that $1 + 2(\cos\frac{2\pi}{2n+1} + \cos\frac{4\pi}{2n+1} + \cdots + \cos\frac{2n\pi}{2n+1}) = 0$, and so $\cos\frac{2\pi}{2n+1} + \cos\frac{4\pi}{2n+1} + \cdots + \cos\frac{2n\pi}{2n+1} = -\frac{1}{2}$.

Problem 12 (2005 AIME I Problem 12 ©MAA)**.**
For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by $S(n) = \tau(1)+\tau(2)+\cdots+\tau(n)$. Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

It is well-known that $\tau(n)$ is odd if and only if n is a perfect square. (Otherwise, we can group divisors into pairs whose product is n.) Thus, $S(n)$ is odd if and only if there are an odd number of perfect squares less than n. So $S(1), S(2)$ and $S(3)$ are odd, while $S(4), S(5), \ldots, S(8)$ are even, and $S(9), \ldots, S(15)$ are odd, and so on.

So, for a given n, if we choose the positive integer m such that $m^2 \leq n < (m+1)^2$ we see that $S(n)$ has the same parity as m.

It follows that the numbers between 1^2 and 2^2, between 3^2 and 4^2, and so on, all the way up to the numbers between 43^2 and $44^2 = 1936$ have $S(n)$ odd. These are the only such numbers less than 2005 (because $45^2 = 2025 > 2005$).

Notice that the difference between consecutive squares are consecutively increasing odd numbers. Thus, there are 3 numbers between 1 (inclusive) and 4 (exclusive), 5 numbers between 4 and 9, and so on. The number of numbers from n^2 to $(n+1)^2$ is

$(n+1-n)(n+1+n) = 2n+1$. Whenever the lowest square beneath a number is odd, the parity will be odd, and the same for even. Thus, $a = [2(1)+1]+[2(3)+1]\dots[2(43)+1] = 3+7+11\dots 87$. $b = [2(2)+1]+[2(4)+1]\dots[2(42)+1]+70 = 5+9\dots 85+70$, the 70 accounting for the difference between 2005 and $44^2 = 1936$, inclusive. Notice that if we align the two and subtract, we get that each difference is equal to 2. Thus, the solution is $|a-b| = |b-a| = |2\cdot 21+70-87| = 25$.

Problem 13 (1988 IMO, proposed by Stephan Beck)**.**
Suppose that a and b are positive integers such that $k = \frac{a^2+b^2}{ab+1}$ is an integer.

Show that k must be a perfect square.

Hint: This problem is a somewhat famous example of the power of the *method of infinite descent*, which focuses on contradicting the existence of a "minimal" example or counterexample. For instance, for this problem, suppose toward a contradiction that the result of the problem is false, and let S denote the (then nonempty) set of pairs (a, b) of positive integers such that $k = \frac{a^2+b^2}{ab+1}$ is an integer but not a perfect square.

We can measure the "size" of a given counterexample by the sum $a + b$ and since $\{a + b : (a, b) \in S\}$ is a set of positive integers, it contains a minimal element. That is, we can find a pair $(a, b) \in S$ with the property that $a + b \le a' + b'$ for any $(a', b') \in S$. To finish the problem, produce a contradiction by producing a strictly smaller counterexample; that is, from this pair (a, b) find some $(a', b') \in S$ with $a' + b' < a + b$.

As an extra hint, we remark that if you relabel as necessary to ensure that $a \ge b$, you will even be able to take $b' = b$ and use the same integer k.

Solution: Let (a, b) be as in the hint, and following the extra hint, relabeling if necessary we assume without loss of generality that $a \ge b$.

Let $k = \frac{a^2+b^2}{ab+1}$, and note that this means that a is a root of the polynomial $p(x) = x^2 - kbx + b^2 - k$. Letting a' denote the other root[1] of p, since the coefficient on x^2 is 1, we know that the sum and product these roots satisfy

$$a + a' = bk, \quad \text{and} \tag{5}$$
$$aa' = b^2 - k. \tag{6}$$

From (5) we see that a' is an integer and from $a'^2 + b^2 = k(a'b+1)$ we see that it must be non-negative (lest $a'b+1$ become non-positive). But we also cannot have $a' = 0$, since (6) would then give $k = b^2$, a perfect square. Thus a' is a positive integer with $k = \frac{a'^2+b^2}{a'b+1}$.

Finally, note that (6) gives

$$a' = \frac{b^2 - k}{a} \le \frac{a^2 - k}{a} = a - \frac{k}{a} < a, \tag{7}$$

so that $a' + b < a + b$, the desired contradiction to the minimality of $a + b$.

[1]Allowing for the moment that this may be a repeat of a, although we will see that they must be distinct.