# Olga Radko Math Circle

by Oleg Gleizer and Olga Radko
with edits from Aaron Anderson and Harris Khan

# Sups, Infs, and Dedekind Completeness

## 1 Completeness of real numbers

We say a set is *countably infinite* if there exists a way to assign each element of that set a natural number, with no repeats and every natural number assigned (in other words, we can "count" the elements of the set). More formally, $A$ is countably infinite if we have a bijective function $f : \mathbb{N} \to A$. A set is *countable* if it is countably infinite or if it is finite. Otherwise, a set is *uncountable*.

**Problem 1** *Prove that $\mathbb{Q}$ is countable, and $\mathbb{R}$ is uncountable.*

The following statement is known as the *Dedekind completeness axiom*.

**Axiom 1** *Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be nonempty subsets such that $a \leq b$ for any $a \in A$ and $b \in B$. Then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for any $a \in A$ and $b \in B$.*

Intuitively, the axiom states that the set of real numbers has no holes. This is not the case for the set of rational numbers $\mathbb{Q}$. Although there exists a rational number in between any two rational numbers, the set of rational numbers has holes in the following sense.

**Problem 2** *Find two nonempty subsets $A, B \subset \mathbb{Q}$ such that $a \leq b$ for each $a \in A$ and $b \in B$, yet there is no rational number $c \in \mathbb{Q}$ such that $a \leq c \leq b$. (Hint: There will be a real number $c$ with this property).*

**Definition 1** *Let us say that a subset $C \subset \mathbb{R}$ is* complete *whenever for every two nonempty subsets $A \subset C$ and $B \subset C$ such that $a \leq b$ for any $a \in A$ and $b \in B$, there exists $c \in C$ such that $a \leq c \leq b$ for any $a \in A$ and $b \in B$.*

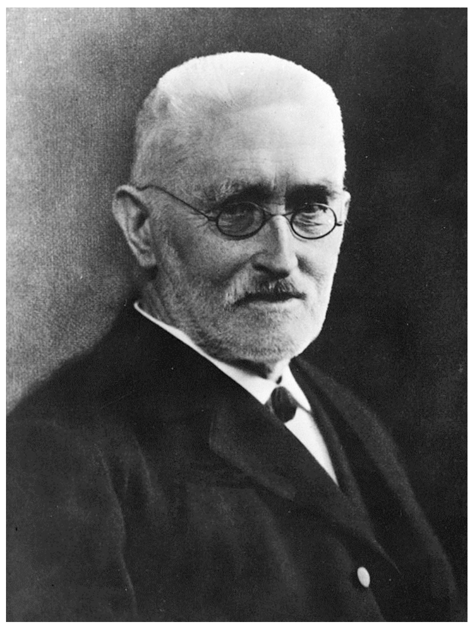
*The Dedekind completeness axiom states that $\mathbb{R}$ is complete, and problem 2 shows that $\mathbb{Q}$ is not.*

**Problem 3** *Note* $\mathbb{Q}$ *is countable and not complete, while* $\mathbb{R}$ *is uncountable and complete.*

*a) Is every subset of* $\mathbb{R}$ *that isn't complete countable?*

*b) Is every complete subset of* $\mathbb{R}$ *uncountable?*

Axiom 1 is named after a German mathematician Julius Wilhelm Richard Dedekind (1831 - 1916), the first person to give an axiomatic definition of real numbers. If you're done with the worksheet, ask your instructors about "Dedekind cuts"!



Richard Dedekind

## 2 Supremum of a Set

Let $A \subset \mathbb{R}$. The number $u \in \mathbb{R}$ is called an *upper bound* of $A$, if $a \leq u$ for any $a \in A$. A set $A$ having an upper bound is called *bounded from above.*

A real number $s$ is called the *supremum of* $A$ a.k.a. the *least upper bound of* $A$ if

- $s$ is an upper bound of $A$ and
- for any upper bound $u$ of $A$, $s \leq u$.

To shorten notations, we write $s = \sup A$.

**Problem 4** *Let $A \subset \mathbb{R}$ be a nonempty subset of real numbers that is bounded from above. Show that the use of "the" to refer to a supremum of $A$ is appropriate. That is, show that if $s, t \in \mathbb{R}$ are both least upper bounds of $A$, then $s = t$.*

**Problem 5** *Let $A = \{a \in \mathbb{R} : 0 \leq a \text{ and } a^2 \leq 2\}$. Prove that $\sup A = \sqrt{2}$.*

**Problem 6** *Let $A = \{(a_n)_{n=1}^{\infty} : a_n = 0.\underbrace{33\ldots3}_{n \text{ times}}\}$. Prove that $\sup A = 1/3$.*

Recall this definition of a limit:

**Definition 2** *The number $A$ is the limit of the sequence $(a_n)_{n=1}^{\infty}$ if for every positive real number $\varepsilon > 0$, there's some natural number $N$ such that for all $n \geq N$, $|a_n - A| < \varepsilon$. If a sequence $(a_n)_{n=1}^{\infty}$ has a limit, we denote it $\lim_{n\to\infty} a_n$.*

**Problem 7** *(a) Let $A \subset \mathbb{R}$ be a nonempty subset of real numbers which is bounded from above. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers entirely contained in $A$ (i.e. $a_n \in A$ for each $n \in \mathbb{N}$). Suppose further that $\lim_{n\to\infty} a_n = L$. Prove that $L \leq \sup(A)$.*

*(b) Using part a, given an alternative argument for problem 6.*

*(c) Let $A \subset \mathbb{R}$ be a nonempty subset of real numbers which is bounded from above. Prove that there is a sequence of real numbers $(a_n)_{n=1}^{\infty}$ entirely contained in $A$ such that $\lim_{n\to\infty} a_n = \sup(A)$. This shows us that the "equal" case from part a can always be obtained.*

**Lemma 1** *A nonempty subset of real numbers bounded from above has the least upper bound.*

**Problem 8** *Prove lemma 1.*

**Lemma 2** *Let* $(a_n)_{n=1}^{\infty}$ *and* $(b_n)_{n=1}^{\infty}$ *be sequences of real numbers. Let* $\lim_{n\to\infty} a_n = L_1$ *and* $\lim_{n\to\infty} b_n = L_2$*. Then the following is true.*

- $\lim\limits_{n\to\infty}(a_n + b_n) = L_1 + L_2$
- $\lim\limits_{n\to\infty}(a_n - b_n) = L_1 - L_2$
- $\lim\limits_{n\to\infty} a_n b_n = L_1 L_2$

*If* $L_2 \neq 0$*, then also*

- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L_1}{L_2}$

**Problem 9** *Prove lemma 2.*

**Problem 10** *One might try to mimic lemma 2 for suprema rather than limits, but it doesn't quite work. Let* $A, B \subset \mathbb{R}$ *be nonempty subsets of real numbers bounded from above.*

- *Let* $A + B$ *be the set of all real numbers of the form* $a + b$ *for* $a \in A$ *and* $b \in B$*. Show* $\sup(A + B) = \sup(A) + \sup(B)$*.*
- *Show that* $\sup(AB)$ *isn't always equal to* $\sup(A)\sup(B)$*. Can you find a condition under which* $\sup(AB) = \sup(A)\sup(B)$*?*
- *What, if anything, can you say about* $\sup(A - B)$*, and* $\sup(A/B)$*?*

Recall a sequence of real numbers $(a_n)_{n=1}^{\infty}$ is called *monotonically increasing* if $m < n \Rightarrow a_m < a_n$.

Now we can prove the lemma from the previous worksheet:

**Lemma 3** *A monotonically increasing sequence of real numbers bounded from above has a limit.*

**Problem 11** Ⓢ *Prove lemma 3.*