Intermediate 2 - Permutations (Part 1)

Edited by Preston Carroll and Richard Yim

13 October 2020

1 Warm-up

You have 100 bags of coins. Each bag has 100 coins, but only one of these bags has special gold coins. Each gold coin weighs 1.01 ounce and all the other mediocre coins weigh 1 ounce each. You also have a weighing scale, but it is running out of battery. You can only use the scale once. How will you identify the bag with the special gold coins?

2 Introduction

We want to ultimately find out whether or not a solution exists for the 15 *puzzle*. What is the 15 puzzle? The 15 puzzle consists of a 4x4 frame randomly filled with 15 squares randomly numbered one through fifteen. The objective of the puzzle is to slide the squares in the puzzle so that they have the following order shown below: The mathematical foundation of the solution relies on the theory of permutations and

Figure 1: Diagram of 15 puzzle.

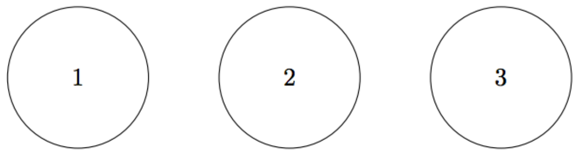
taxicab distance. Permutations not only help unravel the puzzle, but can also be handy in a wide variety of applications from card tricks to probability. Taxicab distance is a way of measuring distance on the plane by going only in the vertical and horizontal direction.

Part 1 of this topic will focus on what permutations are, how they are represented, the multiplication of permutations and the inverse of permutations.

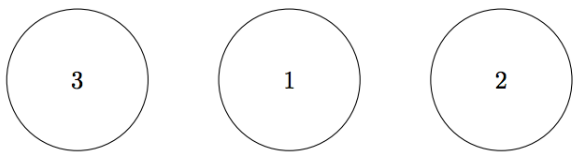
Permutations and Their Representations

Consider a set of marbles numbered 1 through n. Originally, the marbles are lined up in the order given by their numbers.

The following picture shows an example with $n = 3$:



Then the marbles are reshuffled in a different order:



We say that the reshuffled set of marbles is a *permutation* of the first set of marbles. A *permutation* is an operation of the elements of any given set that shuffles the order of the elements.

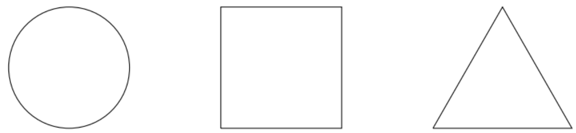
The permutation shown in the example above is represented by the notation below:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

One way to understand the notation above is the following:

- The top row assigns a number to each element in a set
- The bottom row shows how the numbered elements are positioned after we have applied the permutation to the set.

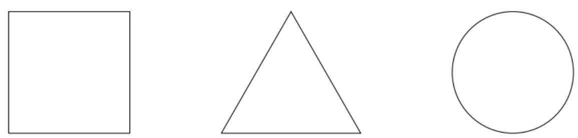
Instead of the numbered marbles, we can reshuffle distinguishable elements of any set. For example, let's consider the following geometric figures rather than the numbered marbles:



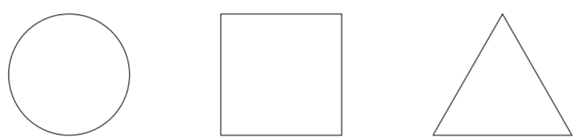
Then the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

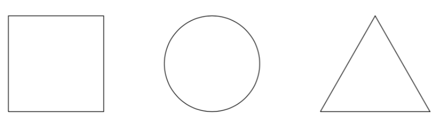
will reshuffle the figures into the following order:



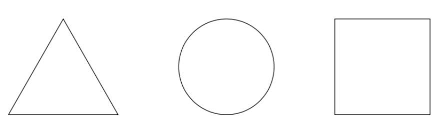
Problem 1. Suppose you're given the following set of geometric figures:



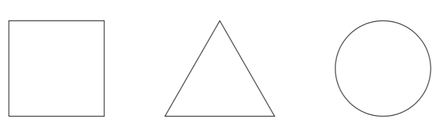
(1) Write down the permutations that correspond to the following pictures:



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

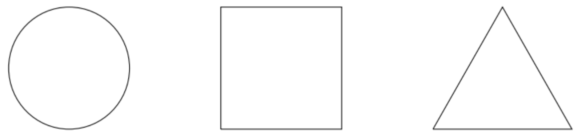


$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

(2) Draw the figures that correspond to the following permutations on the following set:



(a) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Note that the top row is not 1 2 3.

(c) $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

Notice that the last permutation does not reshuffle anything at all. Permutations of this kind are typically denoted as e and are called *trivial*. A trivial permutation is still a permutation, and an important one!

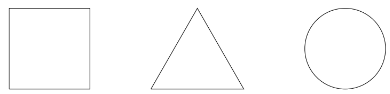
Problem 2. Write down the trivial permutation for $n = 5$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

Multiplying Permutations

It is possible for us to combine, or *multiply*, permutations. The following problem will walk you through how this is done.

Problem 3. Suppose we want to first apply the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and then apply the permutation $\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ on the following set:



(1) First apply the $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ to the set above. Draw the resulting order below:

(2) Now apply $\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ to the ordered set you obtained in (1). Draw the resulting order below:

(3) We say that we have applied the product of δ and σ to the original set. Represent the product of δ and σ as a single permutation by comparing the ordered set you drew after applying both permutations to the original set:

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Note that if we're given a product of permutations such as $\delta \circ \sigma$, the permutation on the right (in this case, σ) is applied to the set first!

Problem 4. Find the permutation of $\sigma \circ \delta$. If needed, use the steps described in the examples above to help you.

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Is $\sigma \circ \delta = \delta \circ \sigma$?

We say that the multiplication of two permutations *commutes* if $\sigma \circ \delta = \delta \circ \sigma$. While some particular permutations may commute, multiplication of permutations in general is not a commutative operation.

Problem 5. Find the product $\delta \circ \sigma$ of the following two permutations:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

Remember to start with the permutation on the right.

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 6. Find the product $\sigma \circ \delta$ of the above permutations.

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

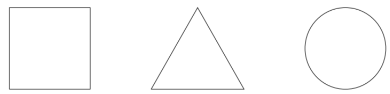
Do the permutations δ and σ commute?

Opposite/Inverse Permutations

Let δ and σ represent permutations. δ is *opposite* to σ if $\delta \circ \sigma = e$. In other words, δ undoes what σ does. Such a permutation is denoted as σ^{-1} and is called the *permutation opposite to sigma* or *sigma inverse*.

Problem 7. Find σ^{-1} for $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

(1) Suppose we are given the following set:



What will the set look like after applying σ to it? Draw the resulting ordered set below.

(2) Taking the second ordered set we drew, we now want to find the permutation σ^{-1} so that it will "undo" what σ did. In other words, we want to find the permutation that will transform the second ordered set back to the original ordered set.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Problem 8. Find σ^{-1} for $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Problem 9. Find σ^{-1} for $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

The above two problems are examples of two different non-trivial permutations that are self inverse. In other words, $\sigma^{-1} = \sigma$.

When dealing with numbers, the only self inverse numbers are -1 and 1 . Unlike numbers, there exist lots of different non-trivial self-inverse permutations.

Problem 10. Given any σ and σ^{-1},

(1) what is $\sigma \circ \sigma^{-1}$?

(2) What is $\sigma^{-1} \circ \sigma$?

(3) Do σ and σ^{-1} commute?

More Notation

You may have noticed that the first line of the notation we have used for writing down permutations is not necessarily needed. The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

means that we shuffle the second element to the first position, the first element to the second position, etc. Without any loss of clarity, we can represent this permutation as

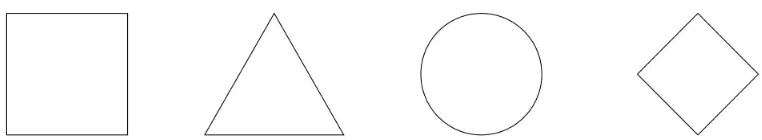
$$\sigma = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}$$

Problem 11. Rewrite the following permutations in our new notation:

(1) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

(2) $\sigma = \begin{pmatrix} 4 & 2 & 3 & 1 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ Hint: Notice that the top row is not 1 2 3 4. Even though the top row is not necessarily constant, we can replace the numbers so that the top row becomes 1 2 3 4.

Problem 12. Apply the following permutations to the sequence of figures shown below:



(1) $\sigma = \begin{pmatrix} 1 & 3 & 2 & 4 \end{pmatrix}$

(2) $\sigma = \begin{pmatrix} 4 & 3 & 1 & 2 \end{pmatrix}$

Problem 13. What is the inverse of the following permutations? You can use the pictures you drew from Problem 13 to help you.

(1) $\sigma = \begin{pmatrix} 1 & 3 & 2 & 4 \end{pmatrix}$

(2) $\sigma = \begin{pmatrix} 4 & 3 & 1 & 2 \end{pmatrix}$

Applying What We Learned to 15 Puzzle

The 15 puzzle was invented by Noyes Palmer Chapman, a postmaster in Canastota, New York, in the mid-1870s. Sam Loyd, a prominent American chess player at the time, had offered $1,000 (about $25,000 of modern day money) for solving the puzzle in the form shown on the picture below:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

We will later learn how to prove that this particular configuration has no solution.

Problem 14. Given that the blank space is 16, write down the permutation corresponding to Loyd's puzzle below. (Assume that the positions in the puzzle are numbered starting from the top left, and go down row by row.)

$$\sigma = (\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad)$$

Problem 15. What is the inverse of the permutation you wrote down above?

$$\sigma^{-1} = (\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad)$$

Math Kangaroo Preparation[1]

(1) Peter, Paul and their grandfather went fishing. During the time that the grandfathers caught 8 fish, Paul caught 4 fish, and Peter caught 7 fish. In one hour, Peter caught 42 fish. How many fish did the three of them catch altogether during that hour?

(2) The number 1999 was multiplied by a number made up of 1999 ones. What is the sum of the digits of this number?

(3) A soccer team consists of 11 players. The average age of the players on a certain team is 22 years old. During a game, one of the players was injured and had to leave the field. The average age of the rest of the players was then 21. How old was the injured player?

(4) The price of theater tickets increased by 40%, but the money earned from ticket sales only increased 26%. How many fewer people went to the theater?

[1] The problems are taken from the 1999 USA Math Kangaroo contest.