

Lesson 2: Vieta's formula

Konstantin Miagkov

October 12, 2018

Definition 1.

We say that x_0 is a root of a function $f(x)$ if $f(x_0) = 0$.

Problem 1.

a) Let $ax^2 + bx + c = 0$ be a quadratic equation. Show that if it has two distinct real roots x_0, x_1 , then $ax^2 + bx + c = a(x - x_0)(x - x_1)$. Hint: consider the difference between $ax^2 + bx + c$ and $a(x - x_0)(x - x_1)$. What degree is it? How many roots does it have?

b) Show that a quadratic equation cannot have more than two distinct real roots.

c) Now suppose that $ax^2 + bx + c$ has exactly one real root x_0 . Show that $ax^2 + bx + c = a(x - x_0)^2$.

Problem 2.

a) [Vieta's formulas] Consider a quadratic equation $ax^2 + bx + c = 0$ with real roots x_0, x_1 . Show that

$$x_0 + x_1 = -b/a$$

$$x_0x_1 = c/a$$

These are called *Vieta's formulas*. You may use the result of problem 1 even if you did not solve it.

b) Given any two real numbers x_0, x_1 with $x_0 + x_1 = u$ and $x_0x_1 = v$, show that both x_0 and x_1 are roots of the quadratic equation $x^2 - ux + v = 0$.

Problem 3.

a) Let x_0, x_1 be roots of a quadratic equation $ax^2 + bx + c = 0$. Find the formula for $x_0^2 + x_1^2$ in terms of a, b, c .

b) Let x_0, x_1 be roots of $x^2 + bx + c = 0$. Find the formula for $x_0^3 + x_1^3$ in terms of b, c .

Problem 4.

Consider a circle whose diameter is the side AB of the triangle ABC . Show that if that circle contains the midpoint of AC , then $\triangle ABC$ is isosceles.

Problem 5.

Let AC be a diameter of a circle, and B be a point on the circle distinct from A and C . Let P be the foot of the perpendicular from A to the tangent to the circle at B . Show that AB is the angle bisector of $\angle PAC$.