Week 1 Handout Solutions - ORMC

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Question 1

$\frac{2012}{2013} < \frac{2013}{2014}$, which follows from

$$\frac{2012}{2013} = \frac{2013-1}{2013} = 1 - \frac{1}{2013}$$
$$\frac{2013}{2014} = \frac{2014-1}{2014} = 1 - \frac{1}{2014}.$$

Question 2

The number 2^{1000} is written down in the decimal system. Would the number of its digits exceed 400? Why or why not?

No. Suppose yes, that the number of digits of 2^{1000} exceeds 400. Then, $2^{1000} \geq 10^{400}$, since 10^{400} is the smallest number whose number of digits exceed 400. Then $2^{1000} \geq 10^{400} > 8^{400} = (2^3)^{400} = 2^{1200} > 2^{1000}$. Contradiction. So no

Question 3

In binary, 2^{1000} does indeed exceed 400 digits, since the binary systems is such that each new place is another power of two, meaning the thousand-and-oneth spot would be reserved for 2^{1000}. There is a single one among the number, as it is of the form

$$10000\ldots000000.$$

Question 4

To get length 8, we draw the segment from 7 to 11 twice, the second one starting where the first one ended.

Then, since we know have a segment of length 8, place the ruler right below it, and mark off a segment of length 1 from 7 to 8. Then add a segment of length 4 from 7 to 11 to obtain a segment of length 5.

Question 5

Three houses A, B, and C, are built along a straight road. You are an engineer commissioned to find a place for a water well W so that the total distance from W to A,B, and C is the shortest possible. Where would you place the well?

We can first define the distance between two houses as their names together, so the distance from house A to house B is AB, and so on. It is clear that if we place the well straight on house B, the total walking distance will be $AB + BC$. If we place it to the right of house B by x units, the total walking distance will be $(AB + x) + (x) + (BC - x) = AB + BC + x$, where $x > 0$, and this is larger than $AB + BC$. If we place it to the left of B by x units, the total walking distance will be $(AB - x) + (x) + (BC + x) = AB + BC + x$, which is also larger. Thus the optimal placement for the well W is right at house B.

Question 6

We use the same logic as Question 5. If we place it on house B, our walk will be $AB + BC + BD$. If we place it on house C, our walk will be $AC + BC + CD = AB + BC + BC + CD = AB + BC + BD$, the same as house B. Then it is clear that if we place it anywhere in between houses B and C, it will also equal the same thing, since moving to the right from house B by x units means both houses A and B move x more, and C and D move x less. If we move anywhere outside the interval from B to C, it is clear that we will have to walk more as well (from Question 5's reasoning), and thus our optimal spot for the well W is anywhere between houses B and C inclusive.

Question 7

We know there are 100 kids in town A and 50 in town B, with a 3 mile road between them. To minimize total walking distance, we place town A at 0 on the x-axis and B on 3, and choose a point $s \in [0, 3]$ to place our school. Then the total walking distance will be

$$D = 100s + 50(3 - s) = 100s + 150 - 50s = 150 + 50s.$$

Thus, this is clearly minimized when s is as small as possible, so we will place the school at $s = 0$, which is just straight in town A.

Question 8

At 1:05, we know that the minute hand is exactly at 1, and the hour hand is 5/60 of the way from 1 to 2. This means that at some time after 1:05, they will overlap (since the minute hand moves quicker but it's behind). To find what time, we have to find a point when the minute hand and hour hand are both the same fraction past 1 and heading toward 2. Let us define the variable $x =$ the number of minutes past 1:05 that we are. Then the minute hand is $x/5$ of the way from 1 to 2, and the hour hand is $\frac{5+x}{60}$ of the way from 1 to 2. Setting these equal, we obtain

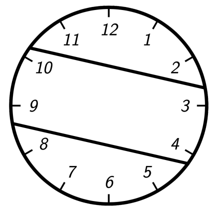
$$\frac{x}{5} = \frac{5+x}{60} \implies 60x = 25 + 5x \implies x = 25/55 \text{ minutes.}$$

Thus, since 25/55 minutes $\approx$ 0.455 minutes $\approx$ 27 seconds, the hour and minute hands will next overlap at about the time 1:05:27.

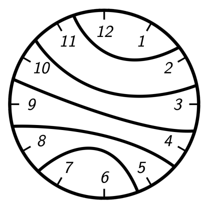
Question 9

To find how many times the hour and minute hand overlap in the day, we will see how many times they overlap in a 12-hour period and double it. Generalizing our last problem, we see that not only do they overlap at 1:05:27 $= 1 + \frac{1}{11}$ hours in, they also overlap at $k + \frac{k}{11}$ hours in, meaning {12:00:00, 1:05:27, 2:10:55, 3:16:22, 4:21:49, 5:27:16, 6:32:44, 7:38:11, 8:43:38, 9:49:05, 10:54:33} is the set of overlaps in a 12-hour period, with cardinality 11. This means per day, the hour and minute hands overlap 22 times.

Question 10



Question 11



Question 12

For our sum, we can note that for any term,

$$\frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thus, when we expand, we will have

$$\begin{aligned}\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \ldots + \frac{1}{(98)(99)} + \frac{1}{(99)(100)} &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{98} - \frac{1}{99} + \frac{1}{99} - \frac{1}{100} \\ &= 1 - \frac{1}{100} = \frac{99}{100}.\end{aligned}$$

Question 13

Given some integer n, since we are working in the decimal system we can write n as

$$n = a_0 + a_1 10 + a_2 10^2 + \ldots + a_k 10^k$$

for some $k \in \mathbb{N}$. Since $10 = 3(3) + 1 \equiv 1 \pmod 3$, it must therefore be that $10^k \equiv 1 \pmod 3$ for any $k \in \mathbb{N}$ as well. Thus when we take congruence modulo 3 on both sides, we "remove" the powers of 10 which are congruent to 1 and obtain

$$n \equiv a_0 + a_1 + a_2 + \ldots + a_k \pmod 3.$$

Since they are congruent, it must be that n is divisible by three when and only when the sum of its digits also is.

Question 14

In a similar manner, we know that $10 = 9 + 1 \equiv 1 \pmod 9$. Thus, given a number

$$n = a_0 + a_1 10 + a_2 10^2 + \ldots + a_k 10^k,$$

we can once again say that

$$n \equiv a_0 + a_1 + a_2 + \ldots + a_k \pmod 9,$$

and thus n is divisible by nine if and only if the sum of its digits also is.

Question 15

We know that $1 \cdot 2 \cdot 3 \cdot \ldots \cdot 100$ is a multiple of 9, since it literally has 9 in it. From 14, we also know that something is a multiple of 9 only when the sum of its digits is a multiple of 9. However, we can apply this again, since its sum of digits is only a multiple of 9 when the sum of the digits of that sum itself is a multiple of 9. We continue this pattern downward (since a number is always larger than its sum of digits), until we must eventually reach 9. For example, taking 10! (which is a multiple of 9), we obtain $10! = 3,628,800 \to 3+6+2+8+8 = 27 \to 2+7 = 9$.

Question 16

Based on the information given, we know that the sum of the digits is $0(100)+1(100)+2(100) = 300$. Since $3+0+0 = 3$, this number is a multiple of 3; however, it is not a multiple of 9 since 3 is not a multiple of 9. This is impossible for a perfect square, since 3 being prime would mean that the square root of this number is also divisible by 3, and thus the square itself is divisible by 9. Thus it cannot be a perfect square.

Question 17

Find the value of $2 \cdot 3$ in the place-value system where $2+3 = 11$.

Here, $2 \cdot 3 = 0$ since $2+3 = 5 \equiv 11$ implies we are working in modular $1,2,3$, or 6 arithmetic, where $2 \cdot 3 = 6 \equiv 0$.

Question 18

$$99 + (9/9) = 99/.99 = 99 + \log_9(9) = (9/.9) \cdot (9/.9) = 100.$$

Question 19

$$(x+y) \times (x-y) = x(x-y) + y(x-y) = x^2 - xy + yx - y^2 = x^2 - y^2.$$

Question 20

Since $999,991 = 1,000,000 - 9 = 1,000^2 - 3^2$, we can rewrite our difference of squares as $1,000^2 - 3^2 = (1000+3)(1000-3) = 1003 \cdot 997 = 999,991$, and so it is not prime.

Question 21

$$\begin{aligned}(x+y)(x^2 - xy + y^2) &= x(x^2 - xy + y^2) + y(x^2 - xy + y^2)\\ &= x^3 - x^2y + xy^2 + yx^2 - xy^2 + y^3\\ &= x^3 + y^3.\end{aligned}$$

Question 22

Given a base b, we know that $1001_b = 1(b^0) + 1(b^3) = b^3 + 1^3$. Using Question 21,

$$b^3 + 1^3 = (b+1)(b^2 - b + 1),$$

and thus 1001_b is composite.