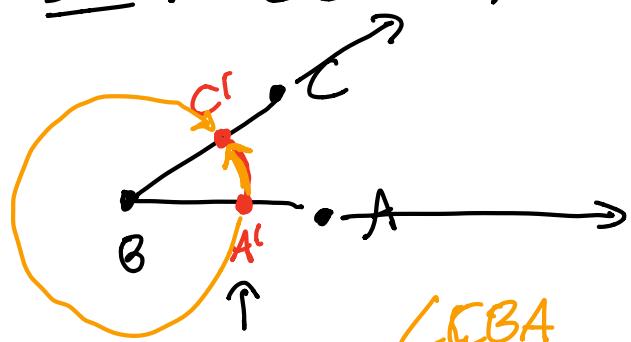


Class 1 - Angle Chasing

EGMO - NOT A REQUIREMENT

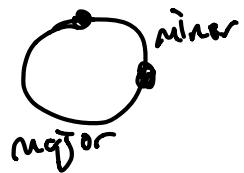
Euclidean Geometry in Mathematical Olympiads

Def. Let A, B, C be points in the plane.



Draw a circle of radius 1 centered at B and suppose ↑ intersects ray \overrightarrow{BA} at A' and \overrightarrow{BC} at C' .

then $\angle ABC := \frac{\text{length of minor arc } A'C}{\text{radius}}$.



→ always positive

- The sum of angles in a triangle is 180°
 $= \pi$

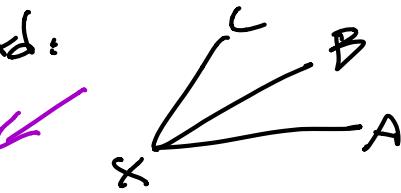
- A right angle is 90°

"Straight angle"

$$\text{---} \geq 180^\circ$$

- Angles add:

$$\begin{aligned} \alpha &= 60^\circ \\ \beta &= 30^\circ \end{aligned}$$



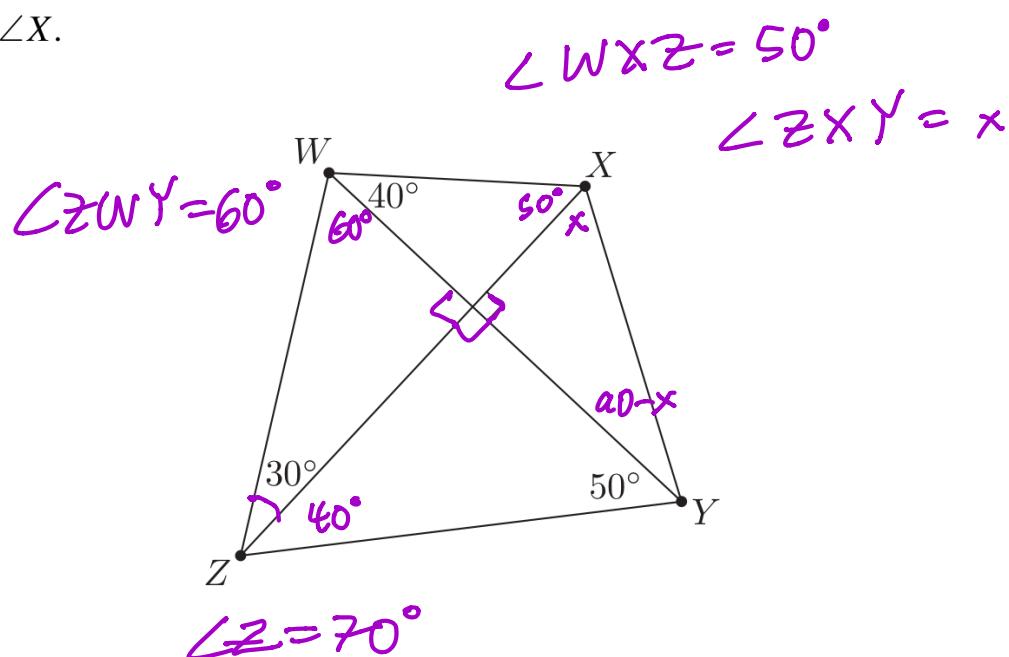
$$\begin{aligned} \angle BAC + \angle CAB \\ = \angle BCA \end{aligned}$$

$$\begin{aligned} \angle CAB + \angle BCA \\ \neq \angle ABC \end{aligned}$$

Example 1.1. In quadrilateral $WXYZ$ with perpendicular diagonals (as in Figure 1.1A), we are given $\angle WZX = 30^\circ$, $\angle XWY = 40^\circ$, and $\angle WYZ = 50^\circ$.

(a) Compute $\angle Z$. $= 70^\circ$

(b) Compute $\angle X$.



Thm (Inscribed Angle theorem)

$$\angle ACO = \alpha \quad \angle BCO = \beta$$

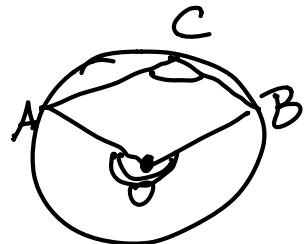
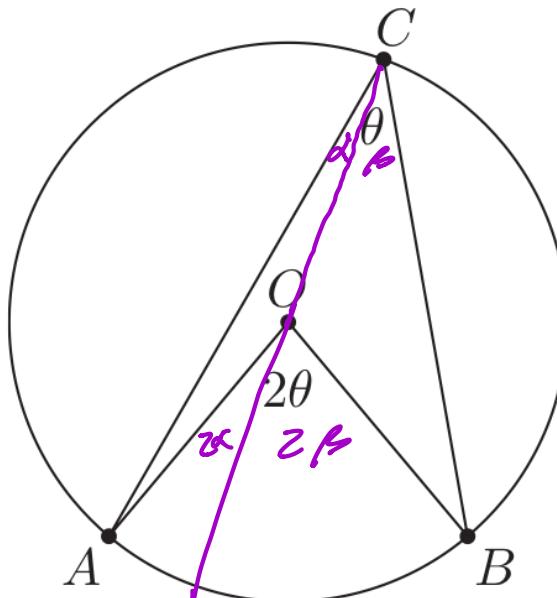
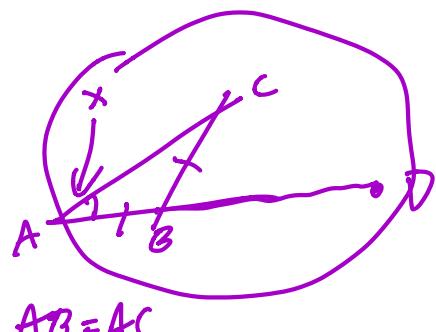
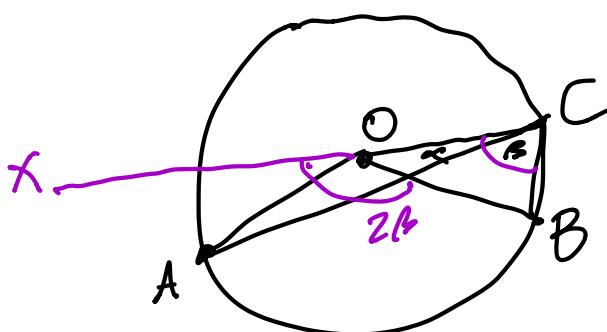


Figure 1.1B. The inscribed angle theorem.

If A, C, B lie on circle w/
center O $\angle ACB =$

$\frac{1}{2}(\angle AOB)$, or $\angle ACB = \frac{1}{2}\widehat{AB}$ where
 \widehat{AB} is the arc not containing C .

Proof.



$$\begin{aligned}\angle ACB &= \beta - \alpha \\ \alpha &= \angle ACO \quad \beta = \angle BCO \\ \hline \angle BOC &= 2\beta\end{aligned}$$

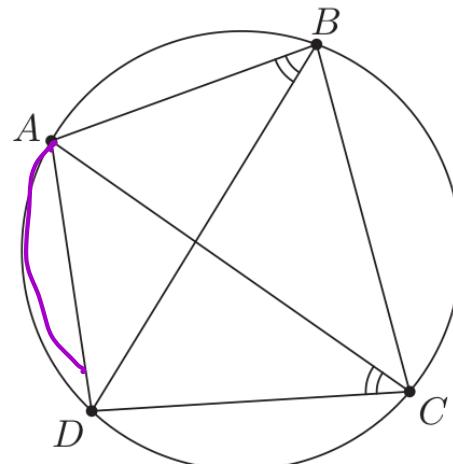
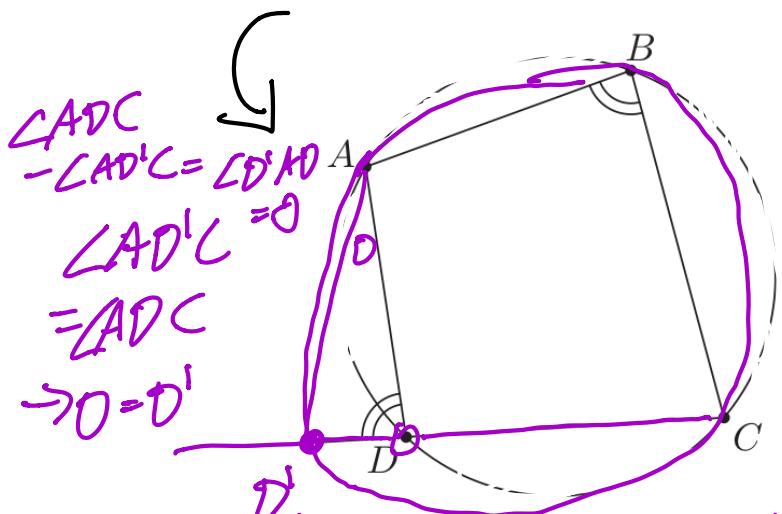
$$\begin{aligned}\angle AOX &= 2\alpha \\ \angle AOB &= \angle BOC - \angle AOX \\ &= 2(\beta - \alpha) \\ &= 2(\angle ACB) \quad \square\end{aligned}$$

Thm. (Cyclic quadrilaterals)

Let A, B, C, D be the vertices of a convex quadrilateral in order. Then the following are equiv:

(1) A, B, C, D all lie on a circle.

(2) $\angle ABC + \angle CDA = 180^\circ$ (3) $\angle ABD = \angle ACD$

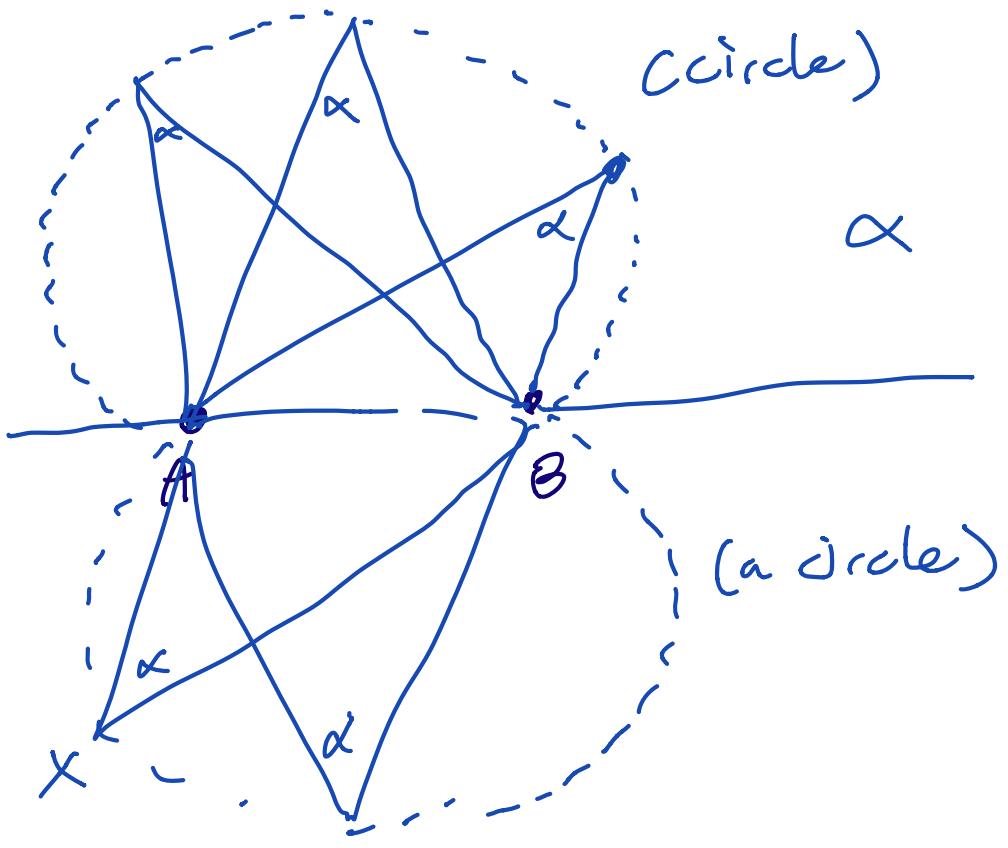


$$\begin{aligned} & \angle ABD \\ & \angle ACD \\ & \approx \frac{1}{2} \angle AOD \end{aligned}$$

A quadrilateral having this property is called cyclic. ^{phantom point}

(1) \rightarrow (2), (1) \rightarrow (3) we Inscribed Angle Thm.

(2) \rightarrow (1) Suppose $ABCD$ satisfies (2).



$$\angle A \times \beta < \alpha$$

$\triangle WPZ \sim \triangle PY$

$\triangle WXP \sim \triangle ZYX$

$WXYZ$ cyclic

$$\frac{PZ}{PW} = \frac{PY}{PX}$$

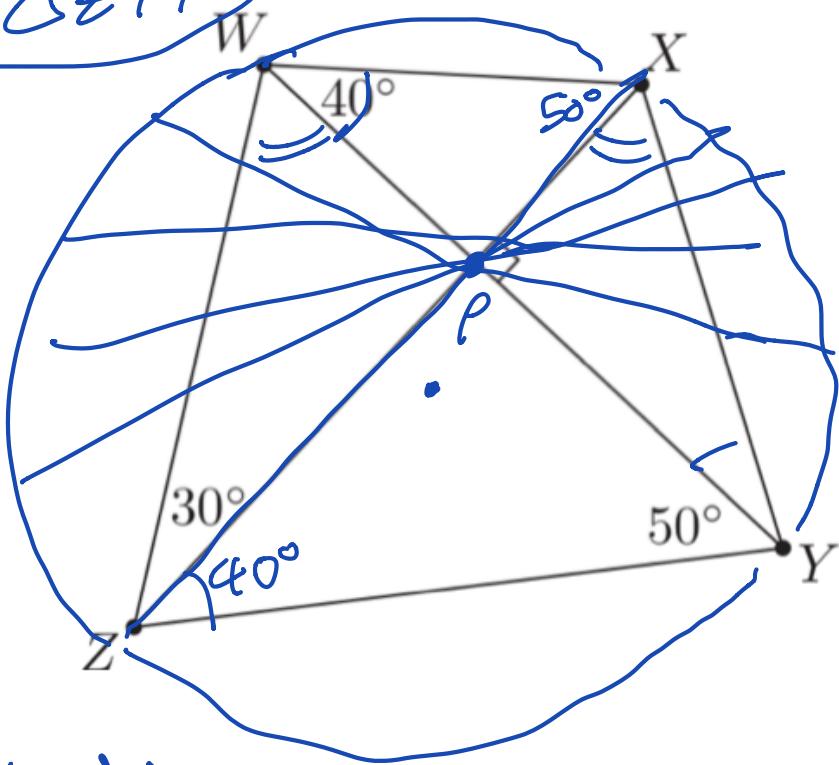
$\angle WZP \neq PY$

$$\angle X - PZ$$

$$= \angle Y - PW$$

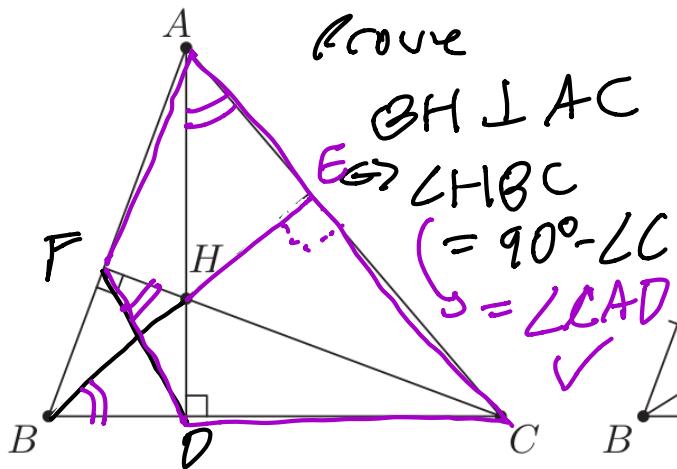
Power of points

$$\angle ZX Y \\ = \angle ZWY$$

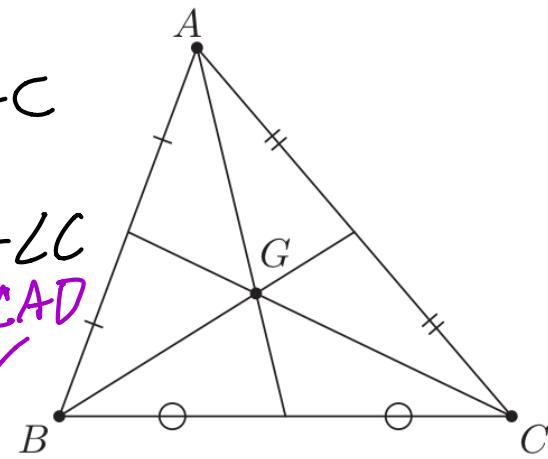


$$\angle Z = 70^\circ \rightarrow \angle X + \angle Z = 180^\circ$$

Orthocenter

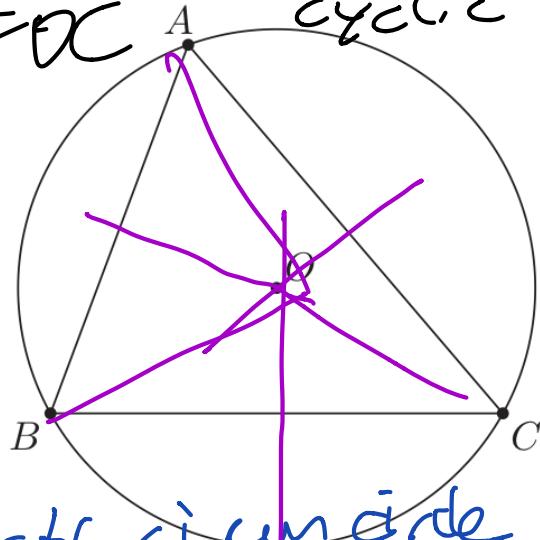


Centroid



$$\angle ADC = \angle AFC$$

AFDC cyclic

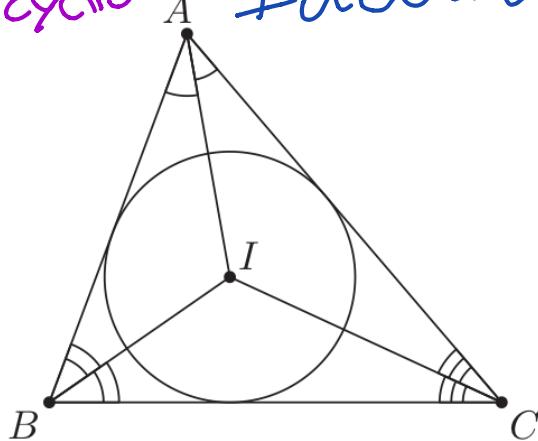


on circumcircle

$$BFHD$$

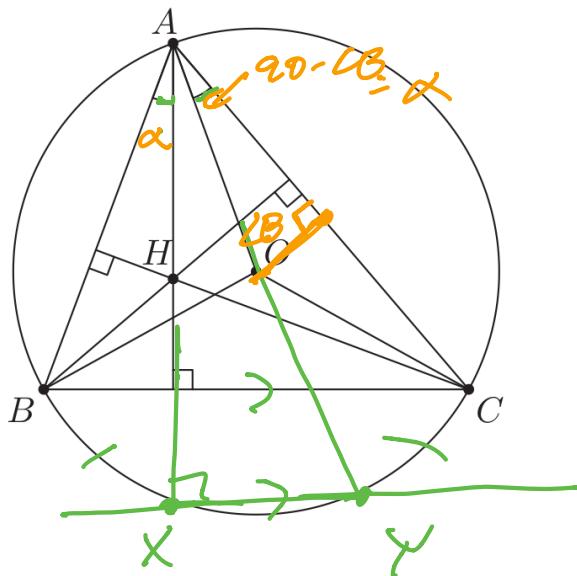
cyclic

Isocenter



incircle
angle bisectors

Problem 1.7. Let O and H denote the circumcenter and orthocenter of an acute $\triangle ABC$, respectively, as in Figure 1.1D. Show that $\angle BAH = \angle CAO$. Hints: 540 373



$H, O \in$
 $B = 90^\circ$ Interior
 $- \alpha$

Ortho = H

Circum = O

Centroid = G

Incircle = I

$\angle CA\bar{Y} = 90^\circ$ since $A\bar{Y}$ is a diameter

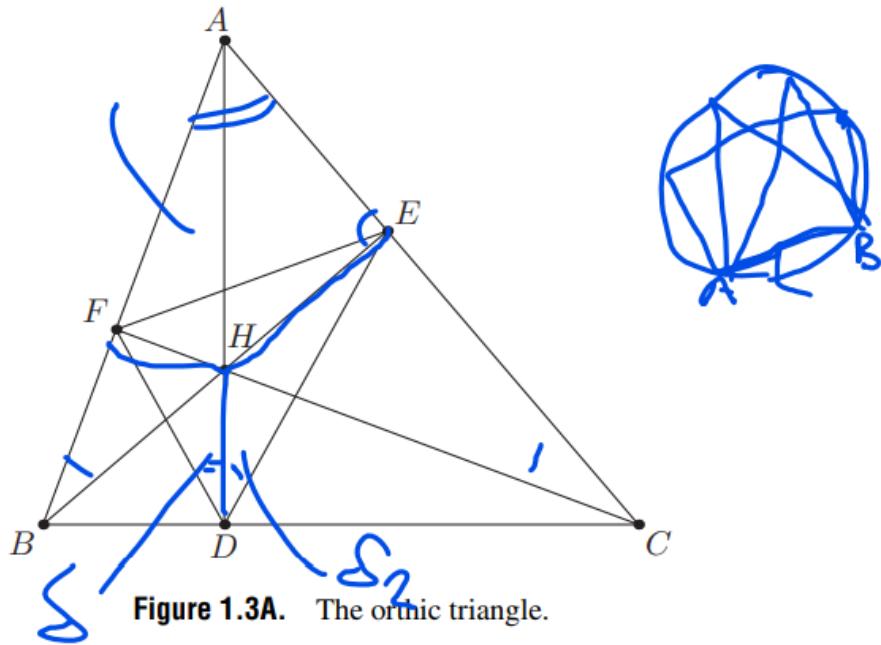
$XY \perp BC$

Diameter

$\widehat{AY} = 180^\circ$

In $\triangle ABC$, let D, E, F denote the feet of the altitudes from A, B , and C . The $\triangle DEF$ is called the **orthic triangle** of $\triangle ABC$. This is illustrated in Figure 1.3A.

$$\begin{aligned} & \text{BFHD} \\ & \angle S = \angle ABE \\ & 90 - A \\ & S_2 = \angle FCA \end{aligned}$$



It also turns out that lines AD, BE , and CF all pass through a common point H , which is called the **orthocenter** of H . We will show the orthocenter exists in Chapter 3.

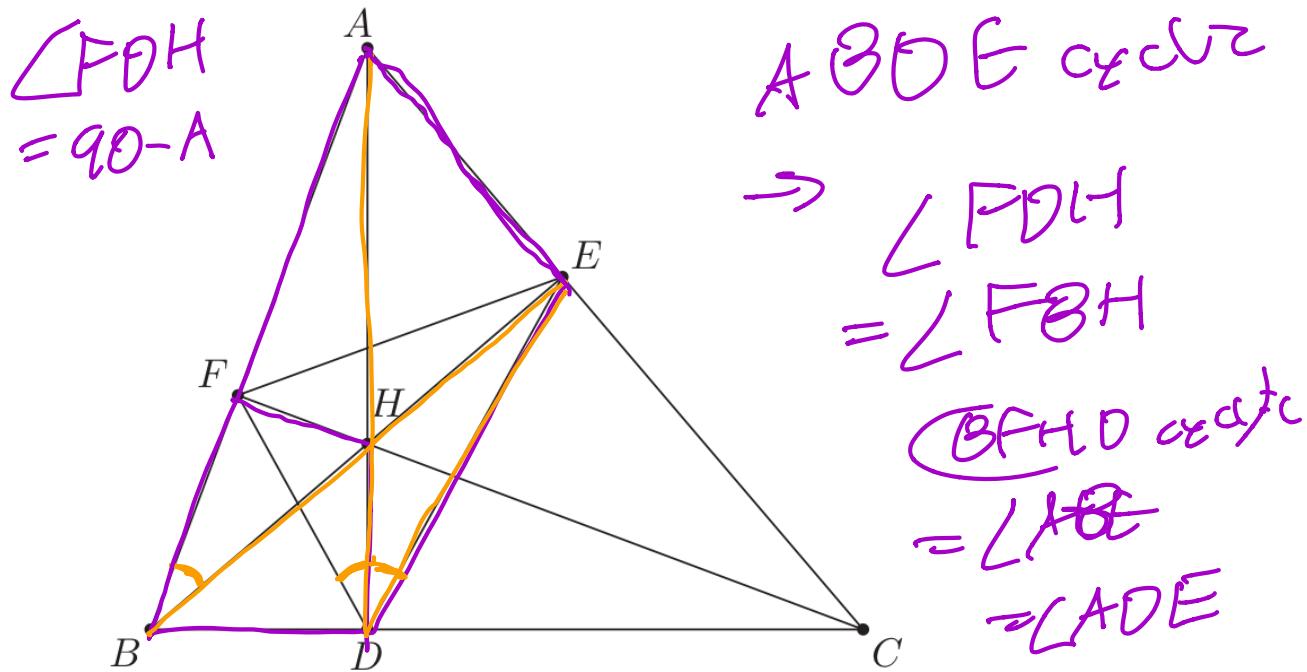


Figure 1.3A. The orthic triangle.

Example 1.13. Prove that H is the incenter of $\triangle DEF$.

$$\angle BXC = 180 - \angle A$$

Code
 $HX = DX$

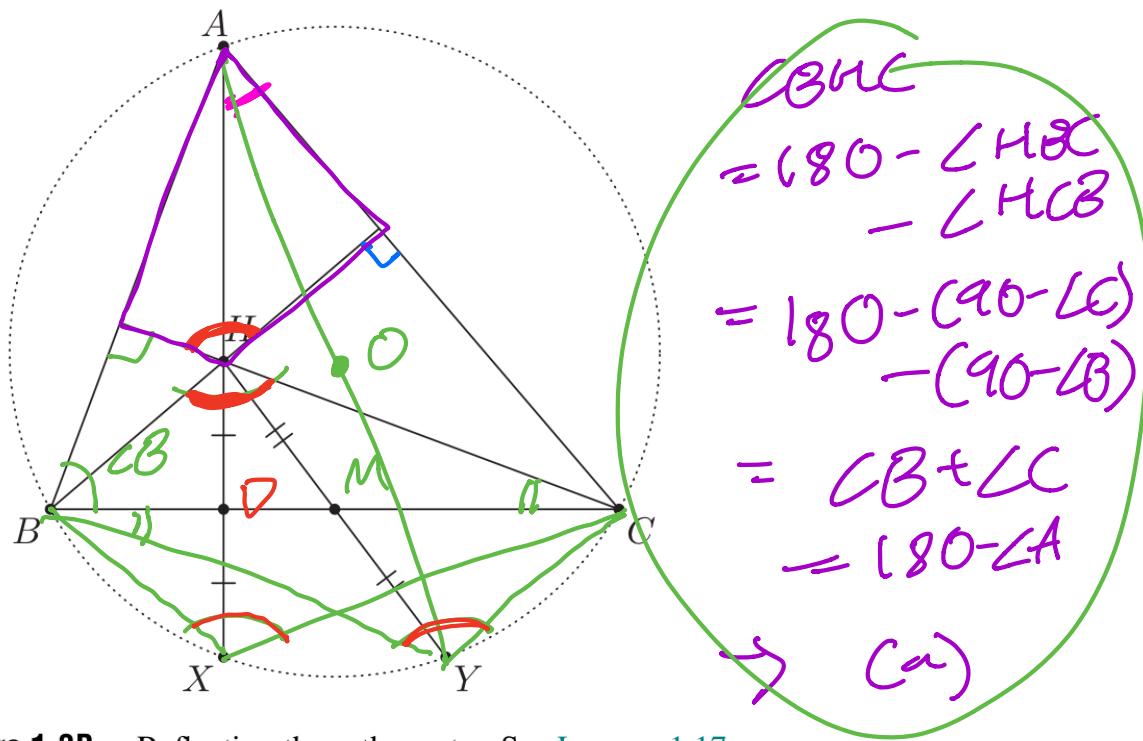


Figure 1.3B. Reflecting the orthocenter. See [Lemma 1.17](#).



Lemma 1.17 (Reflecting the Orthocenter). Let H be the orthocenter of $\triangle ABC$, as in [Figure 1.3B](#). Let X be the reflection of H over \overline{BC} and Y the reflection over the midpoint of \overline{BC} .

- (a) Show that X lies on (ABC) .
- (b) Show that \overline{AY} is a diameter of (ABC) . Hint: [674](#)