COMPETITION-STYLE WARM UP

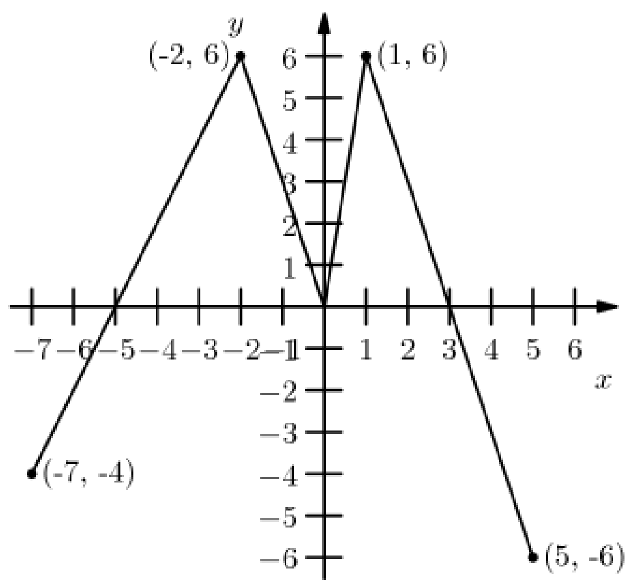
OLGA RADKO MATH CIRCLE
ADVANCED 2
OCTOBER 11, 2020

Problem 1. Can you draw a path on the surface of Rubik's cube (3x3x3 cube) that goes through every single square on the surface? The path should not go through any vertices.

Problem 2 (2010 AIME II Problem 2 ©MAA)**.**
A point P is chosen at random in the interior of a unit square S. Let $d(P)$ denote the distance from P to the closest side of S. Find the probability that $1/5 \leq d(P) \leq 1/3$.

Problem 3 (2002 AMC 12A Problem 16 ©MAA)**.**
Tina randomly selects two distinct numbers from $\{1, 2, 3, 4, 5\}$. Sergio randomly selects one number from the set $\{1, 2, \ldots, 10\}$. What is the probability that Sergio's number is greater than the sum of the two numbers chosen by Tina?

Problem 4 (2002 AMC 12A ©MAA)**.**
If $f : [-7, 5] \to \mathbb{R}$ is the function whose graph is shown below, how many solutions does the equation $f(f(x)) = 6$ have?



Problem 5 (1983 AIME ©MAA)**.**
Find the product of all real solutions to $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$.

Problem 6. Every point in a plane is either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices are the same color.

Problem 7 (2006 AIME I Problem 3 ©MAA)**.**
Find the least positive integer such that when its left-most digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

Problem 8 (2004 Manhattan Mathematical Olympiad)**.**
Seven line segments, with lengths no greater than 10 inches, and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle.
(**Hint:** Order the line segments by increasing length. Given the first two segments, what size would the third have to be to not form a triangle? Continue this way through the list of segments.)

Problem 9 (1988 AIME ©MAA)**.**
Suppose there is a function f defined on the set of ordered pairs (x, y) of positive integers which satisfies

$$f(x, x) = x, \tag{1}$$
$$f(x, y) = f(y, x), \quad \text{and} \tag{2}$$
$$(x + y)f(x, y) = yf(x, x + y). \tag{3}$$

Show that there is only one possible value of $f(14, 52)$ and find it.

Problem 10 (2008 AIME I Problem 11 ©MAA)**.**
Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. How many such sequences have length 14?
(**Hint:** Let a_n and b_n denote, respectively, the number of sequences of length n ending in A and B. Can you relate a_n and b_n to $a_{n-1}, a_{n-2}, b_{n-1}, b_{n-2}$?)

Note: For the next problem, it may be helpful to recall that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and that if $z = e^{\frac{2\pi i}{n}}$, then $1 + z + z^2 + ... + z^{n-1} = 0$.

Problem 11 (D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom)**.** Show that

$$\cos\frac{2\pi}{2n+1} + \cos\frac{4\pi}{2n+1} + \cdots + \cos\frac{2n\pi}{2n+1} = -\frac{1}{2}.$$

(**Hint:** Think about where each term in the sum sits on the circle geometrically and use the equations in the note above the problem.)

Problem 12 (2005 AIME I Problem 12 ©MAA)**.**
For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by $S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$. Let a denote the number of positive integers $n \le 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \le 2005$ with $S(n)$ even. Find $|a - b|$.

Problem 13 (1988 IMO, proposed by Stephan Beck)**.**
Suppose that a and b are positive integers such that $k = \frac{a^2+b^2}{ab+1}$ is an integer.
Show that k must be a perfect square.

Hint: This problem is a somewhat famous example of the power of the *method of infinite descent*, which focuses on contradicting the existence of a "minimal" example or counterexample. For instance, for this problem, suppose toward a contradiction that the result of the problem is false, and let S denote the (then nonempty) set of pairs (a, b) of positive integers such that $k = \frac{a^2+b^2}{ab+1}$ is an integer but not a perfect square.

We can measure the "size" of a given counterexample by the sum $a + b$ and since $\{a + b : (a, b) \in S\}$ is a set of positive integers, it contains a minimal element. That is, we can find a pair $(a, b) \in S$ with the property that $a + b \leq a' + b'$ for any $(a', b') \in S$. To finish the problem, produce a contradiction by producing a strictly smaller counterexample; that is, from this pair (a, b) find some $(a', b') \in S$ with $a' + b' < a + b$.

As an extra hint, we remark that if you relabel as necessary to ensure that $a \geq b$, you will even be able to take $b' = b$ and use the same integer k.