GENERAL APPROACHES: INDUCTION

ORMC, OLYMPIAD GROUP 1, WEEK 1

Induction is a way to solve problems by iterated application of a certain kind of reasoning. It applies to problems of the form "prove the property $P(n)$ for every natural number n." Concretely, a property $P(n)$ can be any statement that depends on n. Here are some basic examples:

$P(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

$P(n) : n^3 + 2n$ is a multiple of 3.

$P(n) : n$ can be written as $3a + 2b$ with a, b natural numbers.

If one can prove the following two facts about P,

- $P(1)$ is true,
- Whenever $P(n)$ is true, $P(n+1)$ is also true,

then it follows that P is true for all natural numbers $n \geqslant 1$. Indeed, from $P(1)$ one gets $P(2)$, from $P(2)$ one gets $P(3)$, and so on. This is induction! Of course, the ideas can be tweaked a bit:

- We can choose to start at a number other than 1. For example, if we prove $P(0)$ instead of $P(1)$, then we actually conclude that $P(n)$ is true for all $n \geqslant 0$ (not just $n \geqslant 1$).
- The 'step' does not have to be one. For example, one may prove that "$P(n)$ implies $P(n+2)$", instead of $P(n+1)$. Then we will find that $P(n)$ is true for all *odd* natural numbers n.
- We can in fact use more information when doing $P(n) \Longrightarrow P(n+1)$. For example, we can assume that $P(1), P(2), \ldots, P(n)$ all hold, not just $P(n)$ (this is called *strong* induction).
- I'm sure you can find more on your own.

Problem 1.

(a) Show that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(b) Show that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(c*) Show that

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Notice how this shows that $(1+2+\cdots+n)^2 = 1^3+2^3+\cdots+n^3$.

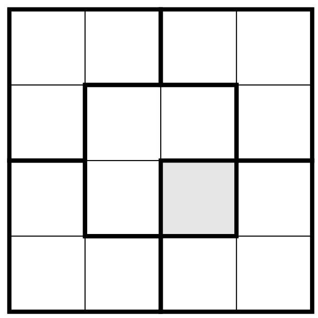
Problem 2.

(a) Show that n^3+2n is always a multiple of 3 for $n \geqslant 0$.

(b*) Show that n^5-n is always a multiple of 30 for $n \geqslant 0$.

Problem 3. We are given n distinct lines in the plane. Show that each connected region cut out by these lines can be colored with black or white, such that any two regions sharing a common side have opposite colors.

Problem 4. Consider a square $2^n \times 2^n$ board, with one 1×1 tile removed. Show that it can be tiled completely with L-shapes of the form ▙. Below is an example for $n=2$:



Problem 5.

(a) Show that $2^n > n^2$ for all $n > 4$.

(b*) Show that $n!\cdot n!\cdot n! > (2n)!$ for $n > 6$. *You may use a calculator for this one.*

Problem 6. We want to show that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \leqslant 2.$$

For this reason, prove that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} \leqslant 2-\frac{1}{n}.$$

Problem 7. Alice and Bob play the following game. Initially, there are n candies on the table. Starting with Alice, they alternatively take either 2 or 3 candies from the table each turn. The player who can't play their turn loses. Show that if n is a multiple of 5 plus 1 or 5, Bob wins, and if n is a multiple of 5 plus 2, 3 or 4 then Alice wins.

Problem 8. Let F_n be the n-th term of the Fibonacci sequence, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Its first few terms are

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

Show that there is a formula

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Problem 9.

(a) Show that $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$.

(b) Show that $5^a \cdot 13^b \cdot 17^c$ can be written as a sum of two squares, for all nonnegative integers a, b, c.

Problem 10. You can also induct on the fact that there are no solutions to a given equation, but you must be careful what variable you induct by. This is one way to look at the method of infinite descent (the quantity that 'descends' should be your induction variable):

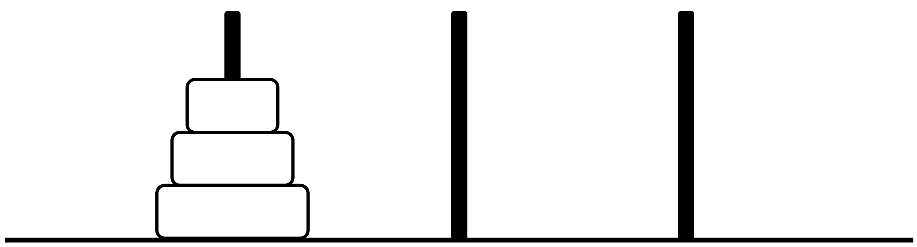
(a) Show that the equation $a^2 + 2b^2 = 2^n$ has no solutions in positive integers a, b, n.

(b) Show that the equation $a^3 + 3b^3 + 9c^3 = 0$ has no solutions in positive integers a, b, c.
Hint: for (b), it's convienent to induct by $n = |abc|$.

Problem *11. We have 3 stacks as in the picture below. The first one contains n disks, arranged in strictly decreasing order of their size, while the last two are empty. The goal is to move the n disks from the first stack to the last stack, subject to the following rules:

- We can only move one disk at a time, from the top of one stack to the top of another stack
- At all times, a larger disk cannot sit on top of a smaller disk.

Show that this can be achieved, for any $n \geqslant 1$.



Problem *12. Show that

$$0 < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \leqslant 1, \qquad \forall n \geqslant 1.$$

You can assume that $\frac{x}{1+x} < \ln(1+x) < x$ for all $x > 0$.

Problem *13. A *graph* is a collection on points in the plane, with some edges between them. Let n be a fixed positive integer, and consider a graph such that each point has at most n neighbors. Show that we can color the points with $n + 1$ colors such that no two neighbors have the same color.