Continued Fractions

A *finite* continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}}$$

where we will assume always that $a_1, \ldots, a_n$ are positive integers. We often use the more compact notation $[a_0, a_1, \ldots, a_n]$.

To compute the continued fraction expansion of a number, for example $3\frac{2}{3}$, we use the following recursive procedure:

- Write $3\frac{2}{3} = 3 + \frac{2}{3}$, separating out the largest whole number possible.
- Rewrite this as $3 + \frac{1}{\frac{3}{2}}$, expressing the non-whole number part $\frac{2}{3}$ as 1 over its reciprocal.
- Repeat the procedure on the denominator $\frac{3}{2}$.

If there is nothing left over after we take away the largest whole number, the procedure terminates. Here is the procedure carried out on $3\frac{2}{3}$:

$$\begin{aligned} 3\frac{2}{3} &= 3 + \frac{2}{3} \\ &= 3 + \frac{1}{\frac{3}{2}} \\ &= 3 + \frac{1}{1 + \frac{1}{2}}. \end{aligned}$$

Exercises

1. Compute a continued fraction expansion for each of the following numbers:

   (a) $\frac{5}{12}$ (b) $\frac{5}{3}$ (c) $\frac{33}{23}$ (d) $\frac{37}{31}$

2. For each continued fraction, write the corresponding number as a reduced fraction:

   (a) $[2, 3, 2]$ (b) $[1, 4, 6, 4]$ (c) $[2, 3, 2, 3]$ (d) $[9, 12, 21, 2]$

Infinite Continued Fractions

The procedure outlined in the previous section may be carried out for any number x: At each step we subtract the largest whole number that we can, keeping the result nonnegative. Then we invert the remainder and repeat.

Example

If $x = \sqrt{2} \approx 1.414...$, we have

$$\begin{aligned}
\sqrt{2} &= 1 + \frac{1}{1+\sqrt{2}} \\
&= 1 + \frac{1}{1 + \left(1 + \dfrac{1}{1+\sqrt{2}}\right)} \\
&= 1 + \frac{1}{2 + \dfrac{1}{1+\sqrt{2}}} \\
&= \ldots
\end{aligned}$$

Exercise

1. Find continued fraction expansions for each irrational number.
   (a) $\sqrt{3}$ (b) $\sqrt{4}$:) (c) $\sqrt{5}$ (d) $\sqrt{6}$ ... (as many as you can!)