Los Angeles Math Circle

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Euler’s number

The goal of this mini-course is to give an accurate construction of the Euler’s number $e$, one of the most fundamental constants in mathematics, physics, economics, and finance. The construction and the applications of the number $e$ to finance and probability are broken into steps and presented as series of problems for students to solve. Harder problems are marked with the red pepper $\bigtriangledown$ sign.

1 Compounded interest

Let $P$ be the primary capital invested at a constant rate $r$ compounded annually. Let $V(t)$ be the value of the investment in $t$ years.

**Problem 1** Derive the formula for $V(t)$.

**Problem 2** Derive the formula for $V(t)$ if the annual rate $r$ is compounded monthly.

**Problem 3** Derive the formula for $V(t)$ if the annual rate $r$ is compounded $n$ times a year, $n \in \mathbb{N}$.

In this worksheet, we will explore what it means to compound interest continuously.

2 Preliminaries on Limits

Let $a_1, a_2, a_3, \ldots$, or alternately, $(a_n)_{n=1}^\infty$, be a sequence of real numbers. The number $u \in \mathbb{R}$ is called an upper bound of $(a_n)_{n=1}^\infty$, if $a_n \leq u$ for any $n \in \mathbb{N}$. A sequence $(a_n)_{n=1}^\infty$ having an upper bound is called bounded from above.
A sequence of real numbers \((a_n)_{n=1}^{\infty}\) is called monotonically increasing if 
\[m < n \Rightarrow a_m < a_n.\]

**Lemma 1** A monotonically increasing sequence of real numbers bounded from above has a unique limit.

Intuitively, a limit of a sequence is a point that the sequence gets closer and closer to. For a few examples of monotonically increasing sequences of real numbers, bounded from above, and their limits, consider the sequence \(-\frac{1}{1}, -\frac{1}{2}, -\frac{1}{3}, \ldots\), with limit 0, and the sequence 3, 3.1, 3.14, 3.141, . . . which limits to \(\pi\).

**Problem 4** Let \(a_1 = 2\), and when \(a_n\) is defined, define \(a_{n+1}\) to be \(2 + \sqrt{a_n}\). Show that the sequence \((a_n)_{n=1}^{\infty}\) is monotonically increasing and bounded above. Find its limit.

**Problem 5** Both assumptions of Lemma 1 are necessary.

Can you come up with an example of a monotonically increasing sequence of real numbers which does not have a limit? How about a sequence of real numbers which is bounded above but does not have a limit?

Now let’s provide a formal definition of the limit of a sequence. If you’re stuck trying to prove something with this formal definition, you can just give an informal explanation, and come back to the problem later.

**Definition 1** The number \(A\) is the limit of the sequence \((a_n)_{n=1}^{\infty}\) if for every positive real number \(\varepsilon > 0\), there’s some natural number \(N\) such that for all \(n \geq N\), 
\[|a_n - A| < \varepsilon.\]
If a sequence \((a_n)_{n=1}^{\infty}\) has a limit, we denote it 
\[\lim_{n \to \infty} a_n.\]

**Problem 6**

- Show that 0 is actually the limit of the sequence \((a_n)_{n=1}^{\infty}\) where \(a_n = -\frac{1}{n}\).
- Show that \(\pi\) is actually the limit of the sequence 3, 3.1, 3.14, 3.141, . . . , where each term gets one more digit of \(\pi\).

**Problem 7** Prove that if a sequence \((a_n)_{n=1}^{\infty}\) has a limit, that limit is unique: If both \(A\) and \(B\) are limits of \((a_n)_{n=1}^{\infty}\), then show \(A = B\).
3 Defining \( e \)

**Problem 8** Recall and prove the binomial identity.

**Problem 9** \( \color{red}{	ext{\(5\)}} \) Prove that

\[
\left(1 + \frac{1}{n}\right)^n < 3 - \frac{1}{n}
\]

(1)

for \( n = 3, 4, \ldots \)

Problem 9 shows that the sequence

\[
e_n = \left(1 + \frac{1}{n}\right)^n, \ n = 1, 2, \ldots
\]

(2)

is bounded from above, \( e_n < 3 \) for \( n \in \mathbb{N} \).

The following very useful statement is known as *Bernoulli inequality*.

\[
(1 + x)^n \geq 1 + nx \text{ for } x \geq -1 \text{ and } n \in \mathbb{N}
\]

(3)

**Problem 10** Use induction to prove \( \color{red}{3} \).

**Problem 11** \( \color{red}{\text{\(5\)}} \) Use Bernoulli inequality to prove that the sequence \( e_n \) defined by (2) is monotonically increasing.

Problems 9, 11 and lemma 1 show that the sequence \((e_n)_{n=1}^{\infty}\) has a limit.

\[
e \overset{\text{def}}{=} \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n
\]

(4)

4 Continuously compounding interest

**Problem 12** Derive the formula for \( V(t) \) if the annual rate \( r \) is compounded continuously.

**Problem 13** Which of the investments described in problems 7, 8 and 12 is a better choice? Why?
5 More about $e$

Problem 14 Similarly, prove the following formula.

$$\lim_{n \to \infty} \left(1 + \frac{1}{n+1}\right)^n = e \quad (5)$$

Problem 15 Prove the following formula.

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1} = e \quad (6)$$

Problem 16 Prove the following formula.

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} \quad (7)$$

Problem 17 Prove the following formula.

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (8)$$

Note that (7) is a particular case of (8) for $x = -1$.

The following very important formula will be proven in a Calculus course.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 2 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \quad (9)$$

Problem 18 Find the first six significant digits of $e$.

Problem 19 If $f : \mathbb{R} \to \mathbb{R}$ is a function, we say that $L$ is a limit of $f$ at $+\infty$ when for every positive real $\varepsilon > 0$, there exists some real number $M$ such that if $x \geq M$, then $|f(x) - L| < \varepsilon$. In that case, we say that $L = \lim_{x \to +\infty} f(x)$.

Prove the following formula.

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e \quad (10)$$

Hint: if $x > 0$, then $\lfloor x \rfloor \leq x \leq [x]$. 
Problem 20 If \( f : \mathbb{R} \to \mathbb{R} \) is a function, we say that \( L \) is a limit of \( f \) at 0 on the right when for every positive real \( \varepsilon > 0 \), there exists some (small) positive real number \( \delta > 0 \) such that if \( 0 < x < \delta \), then \( |f(x) - L| < \varepsilon \). In that case, we say that \( L = \lim_{x \to 0^+} f(x) \).

Prove the following formula.

\[
\lim_{x \to 0^+} (1 + x)^{\frac{1}{x}} = e
\]

6 Euler’s number and probability

Problem 21 A gambler plays a slot machine \( n \) times. Each time, his chance to win is \( p \). What is his chance to win \( k \) times?

Problem 22 A gambler plays 10,000 times a slot machine that pays out one time in 10,000. What is the chance that the gambler loses every bet?

Problem 23 A group of \( n \) people are participating in a gift exchange. Each person puts their name in a hat, and then everyone draws a random name from the hat. For large \( n \), what is the probability that someone draws their own name?