Introduction to Graph Theory

Richard Yim

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Contents

1 Seven Bridges of Konigsberg

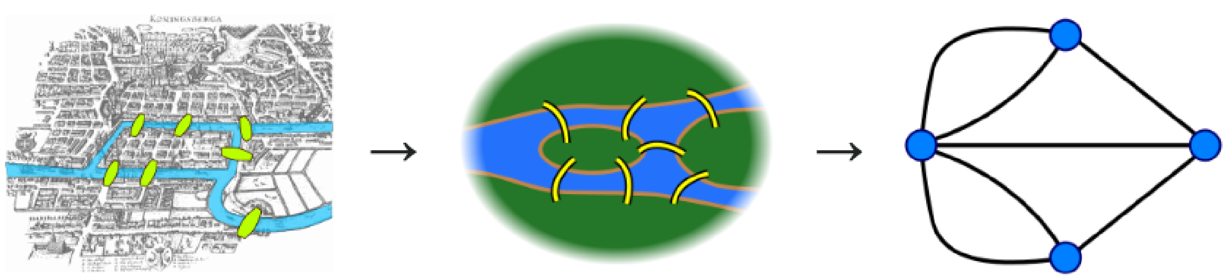


Figure 1: (Figure from https://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg)

Problem: Can we walk through all the seven bridges of Königsberg exactly once?

We can abstract the map of this city into a structure called a *graph*. We represent the landmasses of the city as *vertices*, basically points, and the bridges as *edges*, connections between these points.

This abstraction of the map is what we call positionally invariant. For this problem the size of the vertices, their positions, orientations, distances, are not relevant.

Leonhard Euler (1707-1783) found that in order for such a walk, where we visit each bridge exactly once, an *Eulerian walk*, either all vertices in the graph have even degeree, or exactly two vertices have odd degree.

2 Definitions

Graph The pair of sets (collections of objects) $G = (V, E)$ where V is a collection of vertices and E is a collection of edges between them.

Path - A graph that is a sequence of edges such that each edge ends on the starting vertex of the next edge.

Connected - A graph where there exists a path between any two vertices.

Cycle - A path that begins and ends on the same vertex *without repeating edges.*

Tree - A graph that is connected and has no cycles.

Degree (of a vertex) - The number of edges that meet at a given vertex. (e.g. if 4 edges meet at a vertex v, then we say that, "vertex v has degree 4," and we denote this as $deg(v)$.)

3 Handshake Lemma

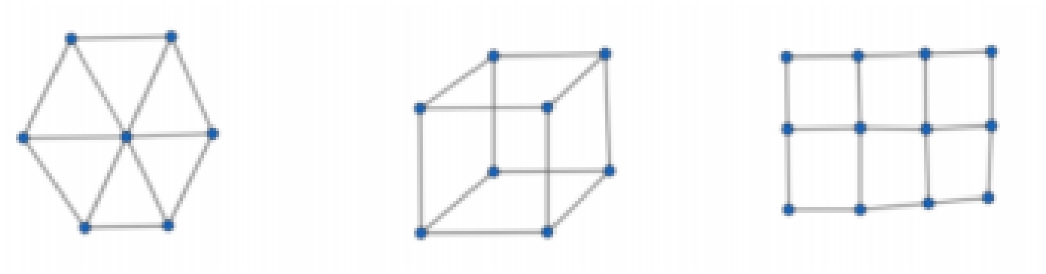
1. Show that the sum of the degrees of all vertices in an graph $G = (V, E)$ is equal to twice the number of edges. In other words, we show that

$$\sum_{v \in V} deg\ (v) = 2|E|$$

2. Is is true that the number of odd degree vertices for any graph is even; why?

4 Trees

1. Show that for any graph $G = (V, E)$ having no cycles satisfying $|V| = |E| + 1$ is a tree.
2. Show that a connected graph G with n vertices is a tree *if and only if* it has $n - 1$ edges.
3. For the first graph, find an induced subgraph that has a vertex with degree 6 and another with degree 3. For the other two graphs is it possible to create subgraph trees where the degrees of all the vertices are either 0 or 1 only?

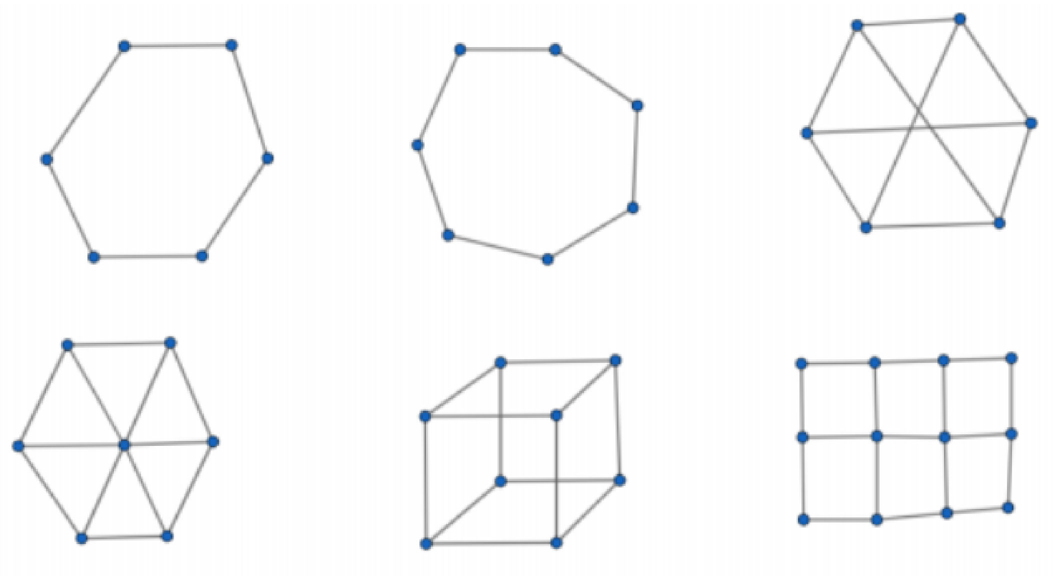


5 Bipartite

1. Let $G = (V, E)$ be a connected unoriented graph such that each cycle has even length. Show that the vertices can be colored with red and blue such that no two neighbors share the same color.

2. Show that a graph is bipartite *if and only if* it does not have odd cycles.

3. Which graphs are bipartite?



6 References

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2. *Invitation to Discrete Mathematics, 2nd Edition* by Jiri Matousek and Jaroslav Nesetril