AMC 8: Miscellaneous Topics

I. Venn Diagrams

Definitions:

A set is simply an unordered collection of elements.

The *cardinality* of a set is the number of elements that the set contains.

The *union* of two sets is a set that contains all elements of the two sets, without duplication.

The *intersection* of two sets is a set that contains all elements that are *common to both sets*.

Venn diagrams are a useful tool to analyze relationships between various sets.

Example Problems:

1. The number of elements in the union of two sets A and B is 49. The cardinality of set A is 27 and the cardinality of set B is 29. What is the number of elements in intersection of the two sets? **7.**

Adding all the elements in the first set and the second set gives us $27 + 29 = 56$, but we know that the total number of elements in both sets is 49. Thus, we have 7 duplicate elements, so the number of elements in the intersection of the two sets is 7.

2. In a town of 351 adults, every adult owns a car, motorcycle, or both. If 331 adults own cars and 45 adults own motorcycles, how many of the car owners do not own a motorcycle? **306**

Adding all the elements in the first set and the second set gives us $331 + 45 = 376$, but we know that the total number of elements in both sets is 351. Thus, we have $376 - 351 = 25$ duplicate elements (the people that own both a car and a motorcycle). Then $331 - 25 = 306$ people own cars but not motorcycles.

3. In a group of 41 children, 24 play soccer, 16 play basketball, and 10 play tennis. Given that the number of children who play soccer and tennis is twice the number of children who play soccer and basketball, and knowing that no children play both tennis and basketball, find:

a) How many children play both soccer and tennis.

b) How many children play both soccer and basketball.

Adding all the elements in the three sets gives us 50, so 9 people play two sports. We know that the number of children who play soccer and tennis is twice the number of children who play soccer and basketball, so we form the equation $2x + x = 9 => x = 3$.

Thus, the number of children that play both soccer and tennis is 6, and the number of children that play both soccer and basketball is 3.

II. Rate & Work

<u>Key formula</u>: $d = v \cdot t$ where d represents a distance, v represents a speed or velocity, and t represents a period of time. Equivalently, one can write $v = \frac{d}{t}$ or $t = \frac{d}{v}$.

<u>Warning</u>: Pay attention to the UNITS of each quantity. For example, if your speed is in miles per hour, make sure that your time is in hours, and your distance is in miles. If that's not the case, you must convert each quantity into the correct unit for the formula to hold.

<u>Example Problems:</u>

4. Isabella had a week to read a book for a school assignment. She read an average of 36 pages per day for the first three days and an average of 44 pages per day for the next three days. She then finished the book by reading 10 pages on the last day. How many pages were in the book? **250**

$$\# \ of \ pages = 3 \times 36 + 3 \times 44 + 10 = 250$$

5.On a trip to the beach, Anh traveled 50 miles on the highway and 10 miles on a coastal access road. He drove three times as fast on the highway as on the coastal road. If Anh spent 30 minutes driving on the coastal road, how many minutes did his entire trip take? **80**

Anh's speed on the access road is $\frac{10mi}{30min} = \frac{20mi}{1hr}$. He travels three times as fast on the highway, so his speed there is 60 mph.

The number of minutes it takes him to drive 50 miles on the highway is: $50mi \times \frac{1hr}{60mi} \times \frac{60min}{1hr} = 50 \ min$

Thus, it will take him $50 + 30 = 80$ minutes for the entire trip.

III. Triangle Congruency

Intuitive Definition: Two geometric figures are congruent if they can be placed exactly on top of each other with all parts lining up perfectly. In other words, they can be transformed into each other by using some combinations of rotations, reflections, and so on.

If two triangles are congruent, then all their sides are equal and all their angles are equal. Conversely, if we are given two triangles with all sides equal and all angles equal, the two triangles are congruent.

How to Prove/Identify Triangle Congruency:

There are several methods to prove that two triangles are congruent.

Congruency by SSS (Side-Side-Side): "If all three sides of one triangle are congruent to all three sides of another triangle, then the two triangles are congruent.

This can be proved rigorously; for now, we can simply realize that knowing all the side-lengths of a triangle will also determine the triangle's angles, fulfilling our definition of triangle congruency.

Congruency by SAS (Side-Angle-Side): "If two sides of a triangle and the angle between them are congruent to two sides of a second triangle and an angle between them, then the two triangles are congruent."

Congruency by ASA (Angle-Side-Angle): "If two angles of a triangle and the side between them are congruent to two angles of a second triangle and an side between them, then the two triangles are congruent."

Congruency by AAS (Angle-Angle-Side): "If two angles of a triangle and a side not between them are congruent to two angles of a second triangle and an side not between them, then the two triangles are congruent.

If we know two angles in a triangle, we automatically know the third one because all the angles must sum to 180 degrees. Thus, it makes no difference if the side is in between the angles or not – we just have different theorems for these two cases.

Congruency by HL (Hypotenuse-Leg): "If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and one leg of another right triangle, then the triangles are congruent."

Explanation: Since these are right triangle, we can find the third side using the Pythagorean theorem. Then we use SSS. It does not matter if the right angle is not between the hypotenuse and the leg, again because we can find all three sides.

Note that we cannot use a "theorem" such as SSA – if we have two sides and angle in common between two triangles, the angle MUST be in between the two sides. The reason behind this is will not fix the third side of the triangle – the angle of that side can still be changed.

IV. Triangle Similarity

We saw before that knowing two triangles are congruent if they have three pairs of congruent sides (SSS). Can we do the same thing for angles? In other words, can we make a new theorem AAA that would state the following: "If all three angles of a triangle are congruent to all three angles of a different triangle, the triangles are congruent?"

It turns out that we can't! Doing so wouldn't fix the length of the sides, only the proportion. We could have one triangle with side-lengths x, y, and z and another triangle with the same angles that has side-lengths $2x, 2y, 2z$.

We can't call these triangles congruent. But there is a mathematical term for such triangles: we can them *similar triangles.*

Definition: If two triangles are similar, then all three pairs of angles are congruent and the triangles have the same ratio of sides.

Conversely, if we are given two triangles and all three pairs of angles are congruent and the triangles have the same ratio of sides, then the triangles are similar.

By the same "ratio of sides" we mean that the respective sides of both triangles are in equal proportion to each other. In the example above, the first triangle had side-lengths x, y, z and the second triangle had side-lengths $2x, 2y, 2z$, keeping a ratio of $2:1$. On the other hand, two triangles could not be similar if the side-lengths of the first triangle were x, y, and z and the side-lengths of the second triangle were $x, 2y$, and $4z$, forexample.

How to Prove/Identify Triangle Similarity:

There are three ways to prove two triangles similar:

Similarity by AA: "If two angles of one triangle are congruent to two angles of a second triangle, then the two triangles are similar."

We only need two angles because we can find the third angle automatically, since the sum of angles in a triangle is known to be 180.

Similarity by Proportions: "If all three sides of one triangle are proportional to all three sides of another triangle, then the two triangles are similar.

Similarity with One Angle and Two Sides: "If two sides of one triangle are proportional to two sides of another triangle, and the angles between the two sides are congruent in both triangles, then the two triangles are similar."

Additionally, note that that two triangle congruency implies triangle similarity.

AMC 8 Problems Involving Triangle Congruency and Similarity

https://artofproblemsolving.com/wiki/index.php/2018_AMC_8_Problems/Problem_20

https://artofproblemsolving.com/wiki/index.php/2011_AMC_8_Problems/Problem_16

https://artofproblemsolving.com/wiki/index.php/2002_AMC_8_Problems/Problem_20