Number Theory

Overview

I. **What is Number Theory?**
II. **Divisibility Tests**
III. **Primes, Prime Factorization, GCD, and LCM**
IV. **Periodicity and Patterns**

I. What is Number Theory

Number Theory → Study of integers (whole numbers)
Algebra → Study of relationships
Combinatorics → Study of counting
(Calculus → Study of change, area)

II. Divisibility Tests

Warm-Up Problem: The 5-digit number 2 0 1 8 U is divisible by 9. What is the remainder when this number is divided by 8? *(Source: 2018 AMC, #7* → Diagnostic Test*)*

Answer: 3

Solution: For a number to be divisible by 9, the sum of its digits must be a divisor of 9.
=> $U = 7$ because, if $U = 7$, then $2 + 0 + 1 + 8 + 7 = 18$. The division tells us that the remainder of 20187 by 8 is 3.

In general…

- For a number to be divisible by 2, the last digit must be divisible by 2.
- For a number to be divisible by 5, the last digit must be divisible by 5.
- For a number to be divisible by 10, the last digit must be divisible by 10.
- For a number to be divisible by 3, the sum of its digits must be a divisor of 3.
- For a number to be divisible by 9, the sum of its digits must be a divisor of 9.

With these basic tests, can we construct new ones?
Example, can we come up with a divisibility test for 6?

Yes! The key is: divisibility by 6 ⇔ divisibility by 2 and divisibility by 3.
We can similarly create more complex divisibility rules superimposing several more basic ones.

III. Primes, Prime Factorization, LCM, GCD

A *prime number* is a number with exactly two divisors.
Ex: 2, 3, 5, 7, …

Prime factorization is the act of decomposing a composite (non-prime) number into its prime factors.

Example/Problem: Find the prime factorization of 126.

Answer: $126 = 2 \cdot 3^2 \cdot 7$

Algorithm/Solution:
By our divisibility test, 126 is divisible by 2. We then calculate 126/2 = 63 and try to find one of its factors.

By our divisibility test, 63 is divisible by 3. We then calculate 63/3 = 21 and try to find one of its factors.

and so on…

We continue this process until arriving at a prime number.

One common use of prime factorization is the computation of LCMs and GCDs

LCM → least common multiple
GCD → greatest common divisor

Example/Problem: What is the GCD of 48 and 56? What is their LCM?

Answer: $\gcd(48, 56) = 8; \operatorname{lcm}(48, 56) = 336$

$$48 = 2^4 \cdot 3^1 \cdot 7^0$$
$$56 = 2^3 \cdot 3^0 \cdot 7^1$$

$$GCD = 2^3 \cdot 3^0 \cdot 7^0$$
$$LCM = 2^4 \cdot 3^1 \cdot 7^1$$

In general…

To calculate GCD, take the smallest prime powers in each column.
To calculate LCM, take the largest prime powers in each column.

IV. Periodicity and Patterns

What is a period? a period sequence? The best way to explain is with an example.

Example: 1, 2, 4, 1, 2, 4, 1, 2, 4, 1, 2, 4….

The above sequence is periodic with a period (1, 2, 4)

So what is a period? A period is a motif/pattern that repeats over and over again. But why is this interesting for number theory? Again, an example is most insightful

Example: The powers of 2

2
4
8
16
32
64
128
256
512
1024

The last digits of the powers of 2 form a periodic sequence! Noticing such patterns can help us solve problems that, at first glance, seem infallible.

Problem: What is the last digit (units digit) of 2^{1000}?

Answer: 6

Solution: According to the above pattern, the last digit of 2^n with n divisible by 4 is 6. Since 1000 is divisible by 4, the answer must be 6.