Number Theory

Overview

I. What is Number Theory?
II. Divisibility Tests
III. Primes, Prime Factorization, GCD, and LCM
IV. Periodicity and Patterns

I. What is Number Theory

Number Theory → Study of integers (whole numbers)
Algebra → Study of relationships
Combinatorics → Study of counting
(Calculus → Study of change, area)

II. Divisibility Tests

Warm-Up Problem: The 5-digit number 2018U is divisible by 9. What is the remainder when this number is divided by 8? (Source: 2018 AMC, #7 → Diagnostic Test)

Answer: 3

Solution: For a number to be divisible by 9, the sum of its digits must be a divisor of 9.
=> U = 7 because, if U = 7, then 2 + 0 + 1 + 8 + 7 = 18. The division tells us that the remainder of 20187 by 8 is 3.

In general...

• For a number to be divisible by 2, the last digit must be divisible by 2.
• For a number to be divisible by 5, the last digit must be divisible by 5.
• For a number to be divisible by 10, the last digit must be divisible by 10.
• For a number to be divisible by 3, the sum of its digits must be a divisor of 3.
• For a number to be divisible by 9, the sum of its digits must be a divisor of 9.

With these basic tests, can we construct new ones?
Example, can we come up with a divisibility test for 6?

Yes! The key is: divisibility by 6 ⇔ divisibility by 2 and divisibility by 3.
We can similarly create more complex divisibility rules superimposing several more basic ones.
III. Primes, Prime Factorization, LCM, GCD

A *prime number* is a number with exactly two divisors.
Ex: 2, 3, 5, 7, ...

*Prime factorization* is the act of decomposing a composite (non-prime) number into its prime factors.

**Example/Problem:** Find the prime factorization of 126.

Answer: \(126 = 2 \cdot 3^2 \cdot 7\)

Algorithm/Solution:
By our divisibility test, 126 is divisible by 2. We then calculate \(126/2 = 63\) and try to find one of its factors.

By our divisibility test, 63 is divisible by 3. We then calculate \(63/3 = 21\) and try to find one of its factors.

and so on...

We continue this process until arriving at a prime number.

**One common use of prime factorization is the computation of LCMs and GCDs**

**LCM** → least common multiple
**GCD** → greatest common divisor

**Example/Problem:** What is the GCD of 48 and 56? What is their LCM?

Answer: \(\gcd(48, 56) = 8; \ lcm(48, 56) = 336\)

\[
\begin{align*}
48 &= 2^4 \cdot 3^1 \cdot 7^0 \\
56 &= 2^3 \cdot 3^0 \cdot 7^1
\end{align*}
\]

\[
\begin{align*}
\text{GCD} &= 2^3 \cdot 3^0 \cdot 7^0 \\
\text{LCM} &= 2^4 \cdot 3^1 \cdot 7^1
\end{align*}
\]

**In general...**

To calculate GCD, take the smallest prime powers in each column.
To calculate LCM, take the largest prime powers in each column.
IV. Periodicity and Patterns

What is a period? a period sequence? The best way to explain is with an example.

Example: 1, 2, 4, 1, 2, 4, 1, 2, 4, 1, 2, 4....

The above sequence is periodic with a period (1, 2, 4)

So what is a period? A period is a motif/pattern that repeats over and over again. But why is this interesting for number theory? Again, an example is most insightful

Example: The powers of 2

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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>32</td>
<td>64</td>
<td>128</td>
<td>256</td>
</tr>
<tr>
<td>512</td>
<td>1024</td>
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The last digits of the powers of 2 form a periodic sequence! Noticing such patterns can help us solve problems that, at first glance, seem infallible.

Problem: What is the last digit (units digit) of $2^{1000}$?

Answer: 6

Solution: According to the above pattern, the last digit of $2^n$ with $n$ divisible by 4 is 6. Since 1000 is divisible by 4, the answer must be 6.