Los Angeles Math Circle: Induction 2 (Reading!)

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1 Recap: What is Induction?

Let's look again at the very basic problem of finding out whether or not

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

is true for all whole numbers that are not negative.

Sure, many of you said the following: "Well, let's pick some random n, say $n = 5$. Let's add 1 through 5. Then let's calculate $\frac{5(5+1)}{2}$. Oh! we notice that these are the same! Okay, the formula obviously holds for $n = 5$. Let me try another n..."

And so on and so forth a lot of you computed say 3 or 4 more examples and concluded that that was enough to show that the formula above holds *for all* whole numbers that are not negative, or nonnegative integers.

But this is very flawed! It does not abstractly explain and *argue* why the above formula is correct for any *arbitrary* n; the above answer that many of you made only *observes* the formula holding true for specific n. Mathematics is not about computing and calculating it is about making logical arguments. The above problem of verifying the formula can only be correctly answered as follows... This is using the method of proof called simple mathematical induction:

1. First, we verify a base case. The simplest case that we can choose to verify the above formula is using $n = 0$. Obviously, summing 0 to 0, is 0 — on the left hand side of the above formula. Additionally,

   $$\frac{0 \cdot (0+1)}{2} = 0$$

   Clearly, the base case of the above formula is true. I repeat the main purpose of our problem above: we are aiming to show that the above formula holds FOR ALL nonnegative integers.

2. Now, instead of verifying that the next nonegative integer, 1, also holds for the above formula, we instead *abstractly* suppose that some *arbitrary* n satisfies the above formula. More precisely, we make a *hypothesis* and suppose that the formula is indeed true for some arbitrary n. Namely,

   $$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

   is hypothetically true. But *why* are we making an inductive hypothesis? *How* is this relevant to the problem? Well, what a lot of you have done is chosing each specific $n = k$ value and gone on with computing and applying the above formula. If the above formula is true, like a proper hypothesis, then it should be true, that the following integer $n + 1$ satisfies the above formula. Again, we are aiming to show that the above formula, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ is true FOR ALL nonnegative integers.

3. Now we use our inductive hypothesis and show that if it really is the case that some arbitrary n satisfies the above formula, then its succeeding number, $n + 1$ will satisfy the above formula as well, or

$$1 + 2 + \cdots + n + (n + 1) = \frac{(n + 1)((n + 1) + 1)}{2}$$

is true as well. Well, we now use our inductive hypothesis from (2), and we can write the following

$$\begin{aligned}(1 + 2 + \cdots + n) + (n + 1) &= \left(\frac{n(n + 1)}{2}\right) + (n + 1) = \frac{n(n + 1) + 2(n + 1)}{2} \\ &= \frac{n^2 + 3n + 2}{2} = \frac{(n + 1)(n + 2)}{2} \\ &= \frac{(n + 1)((n + 1) + 1)}{2}\end{aligned}$$

Clearly, we have verified our induction step in showing that the successor to some arbitary nonnegative integer n, namely $n + 1$, does indeed satisfy the formula, as it should appear if it were hypothetically true that the formula holds for n. This completes the proof! We now summarize the steps and why each of them are important.

2 Summary

1. Base Case: the base case actually verifies that some "base" scenario, or an actual and most basic example, holds true for the formula. It is serves a sanity check, and a foundation for showing that successive integers are also true because if this base case were not true, then immediately the statement *for all* is false.

2. Inductive Hypothesis: this hypothesis makes the supposition that the above formula is true for some n. This is important because we can abstractly represent all possible integers with this singular identity of n, without wasting resources on computing each case of the formula until infinity.

3. Induction Step: Using our predecessor, our inductive hypothesis, we show that the successor is true. The idea being that "the next number" holding for the formula *depends* on the previous number satisyfing the formula. This induction step works for the above proof because there is some underlying dependence on a "sequence" holding true for our problem.

3 Conclusion

The above version of proof by induction is a very specific type of proof by induction. There are many other statements, lemmas and theorems that use proof by induction, but not all. A lot of the mathematical statements and claims that use proof by induction do so if they include some description about being true *for all* and some possible dependence on sequences (for example, our above formula has some sequential dependence being the sum of the sequence 1 through n).