Math Circles Intermediate 2A - Induction 2

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Problem 1

We define the Fibonacci Sequence recursively as follows, where F_n denotes the n^{th} Fibonacci number and $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. We expand this out and see that the first few terms of the sequence are $0, 1, 1, 2, 3, 5, 8, 13, etc.$ We can prove many properties about the Fibonacci Sequence by induction, here are a few.

a. $F_{n+1} < (\frac{7}{4})^n$ for all $n \geq 1$.

b. $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$ for all $n \geq 1$.

c. $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$ for all $n \geq 1$.

d. F_{2n} is divisible by F_n for all $n \geq 1$. See if you can generalize this further.

hint: Use the fact that $F_{n+m} = F_{n+1}F_m + F_{m-1}F_n$

e. $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ for all $n \geq 1$.

Problem 2

Show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}$$

for all positive integers.

Problem 3

Show that

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \ldots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

for all $n \geq 2$.

Problem 4 Show that

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$$

for all $n \geq 1$.

hint: think about what the sum of the first n numbers means. We proved it last time.