COMPETITION PROBLEMS

LOS ANGELES MATH CIRCLE
ADVANCED 2
JUNE 7, 2020

Instructions.
(1) Unless specifically requested in the problem, you are not required to prove your answers to receive credit.
(2) You may consult the worksheets from this quarter to remind yourself of any definitions.
(3) All other resources (e.g. calculators, WolframAlpha) are not allowed.
(4) If you have any questions about what facts you can assume, ask an instructor.

1. Fixed points

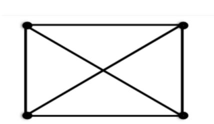
Problem 1 (3 points)**.** Let (X, d) be a metric space with at least two points, and with d the discrete metric. That is, $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1$ if $x \neq y$. Describe all contractions $f : X \to X$ and prove your assertions.

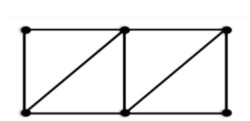
Problem 2 (3 points)**.** Show any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. Can it have two? Prove your assertions.

Problem 3 (4 points)**.** Let $f : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ be the function given by $f(x, y) = (\frac{y^3}{3} - \frac{y}{2} + 1, \frac{x^3}{3} - \frac{x}{2} + 1)$. Prove that f has a unique fixed point.

2. Graph theory, Sperner's lemma, and Brouwer fixed point theorem

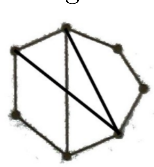
Problem 4 (1 point each)**.** Decide if the following graphs have Eulerian paths. Which of the graphs have Eulerian circuits?

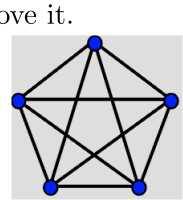
(a) 

(b) 

(c) 

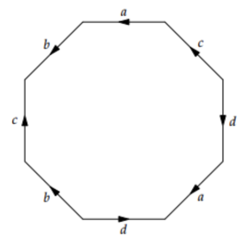
Problem 5 (1 point each)**.** Decide if the following graphs are planar. If so, draw the graph in the plane with no edge crossings. If not, prove it.

(a) 

(b) 

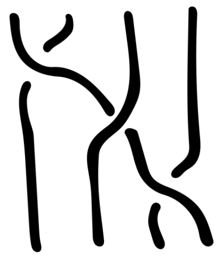
Problem 6 (2 points)**.** Does the set $[0, 1)$ have the Brouwer fixed point property? If so, prove your claim. If not, find a continuous function $f : [0, 1) \to [0, 1)$ which does not have a fixed point.

Problem 7 (2 points)**.** Compute the Euler characteristic of the following polygon with the edges identified as shown.

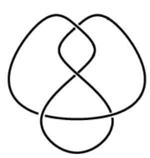
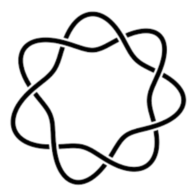


3. Knots and quandles

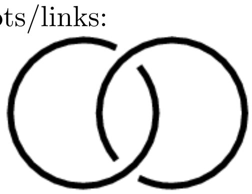
Problem 8 (2 points)**.** Find the inverse of the following braid:



Problem 9 (1 point each)**.** Decide if the following knots are tricolorable:

(a)  (b)  (c) 

Problem 10 (2 points each)**.** Calculate the Kauffman bracket (in terms of A, B, and d) of the following knots/links:

(a)  (b) 

Problem 11 (2 points)**.** Classify all quandles of two elements up to relabeling.

4. Game theory

Problem 12 (2 points)**.** In the following zero-sum game, Player 1 can play U or D while Player 2 can play L or R. Determine what each player chooses if both players play rationally.

	L	R
U	1	1
D	1	-2

Problem 13 (2 points)**.** If both players follow play maximin strategies, what will the outcome of the game be?

	L	R
U	1	2
D	1	-2

Problem 14 (2 points)**.** Consider the following game.

	L	R
U	2,1	0,0
D	0,0	1,2

(a) Find the Pareto optimal outcomes.
(b) Find a mixed Nash equilibrium.

5. Miscellaneous

Problem 15 (3 points)**.** Find the maximum value of the expression

$$H(x,y,z) \;=\; x\log(1/x) + y\log(1/y) + z\log(1/z)$$

subject to the contraints

$$x, y, z \geq 0, \qquad x + y + z = 1.$$

Problem 16. Let A be a subset of the natural numbers $\mathbb{N}$. The **natural density** of A is defined as

$$\delta(A) \;:=\; \lim_{N\to\infty} \frac{|A \cap [1,N]|}{N}$$

if the limit exists. For each of the following, decide if the natural density exists and if so, calculate it.

(a) (1 point) $A = \{2, 4, 6, 8, \dots\}$
(b) (1 point) $A = \{n^2 : n \in \mathbb{N}\}$
(c) (2 points) $A = \{n : n \text{ contains a 9 in its decimal expansion}\}$
(d) (2 points) $A = \{n : \lfloor \log_2 n \rfloor \text{ is even}\}$
(e) (5 points) $A = \{n : n \text{ is squarefree}\}$ (a number n is said to be **squarefree** if n is not divisible by p^2 for any prime p)

Problem 17 (3 points)**.** Calculate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

Problem 18. (2 points) Which is bigger, e^π or π^e? Prove your answer (NO CALCULATORS) (obviously)

Problem 19. (5 points) Does the series

$$\sum_{p \text{ prime}} \frac{1}{p}$$

converge or diverge? Prove your answer.

Problem 20. (3 points) Calculate

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

(Hint: polar coordinates)