

# COMPETITION PROBLEMS

LOS ANGELES MATH CIRCLE  
ADVANCED 2  
JUNE 7, 2020

## Instructions.

- (1) Unless specifically requested in the problem, you are not required to prove your answers to receive credit.
- (2) You may consult the worksheets from this quarter to remind yourself of any definitions.
- (3) All other resources (e.g. calculators, WolframAlpha) are not allowed.
- (4) If you have any questions about what facts you can assume, ask an instructor.

## 1. FIXED POINTS

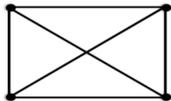
**Problem 1** (3 points). Let  $(X, d)$  be a metric space with at least two points, and with  $d$  the discrete metric. That is,  $d(x, y) = 0$  if  $x = y$ , and  $d(x, y) = 1$  if  $x \neq y$ . Describe all contractions  $f : X \rightarrow X$  and prove your assertions.

**Problem 2** (3 points). Show any continuous function  $f : [0, 1] \rightarrow [0, 1]$  has a fixed point. Can it have two? Prove your assertions.

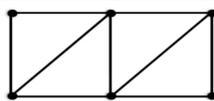
**Problem 3** (4 points). Let  $f : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$  be the function given by  $f(x, y) = (\frac{y^3}{3} - \frac{y}{2} + 1, \frac{x^3}{3} - \frac{x}{2} + 1)$ . Prove that  $f$  has a unique fixed point.

## 2. GRAPH THEORY, SPERNER'S LEMMA, AND BROUWER FIXED POINT THEOREM

**Problem 4** (1 point each). Decide if the following graphs have Eulerian paths. Which of the graphs have Eulerian circuits?



(a)



(b)

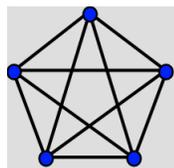


(c)

**Problem 5** (1 point each). Decide if the following graphs are planar. If so, draw the graph in the plane with no edge crossings. If not, prove it.



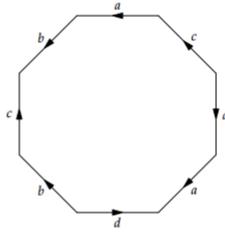
(a)



(b)

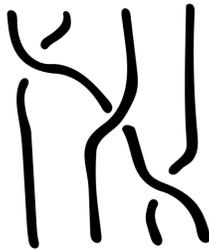
**Problem 6** (2 points). Does the set  $[0, 1)$  have the Brouwer fixed point property? If so, prove your claim. If not, find a continuous function  $f : [0, 1) \rightarrow [0, 1)$  which does not have a fixed point.

**Problem 7** (2 points). Compute the Euler characteristic of the following polygon with the edges identified as shown.



### 3. KNOTS AND QUANDLES

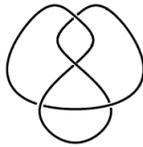
**Problem 8** (2 points). Find the inverse of the following braid:



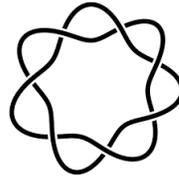
**Problem 9** (1 point each). Decide if the following knots are tricolorable:



(a)



(b)



(c)

**Problem 10** (2 points each). Calculate the Kauffman bracket (in terms of  $A$ ,  $B$ , and  $d$ ) of the following knots/links:



(a)



(b)

**Problem 11** (2 points). Classify all quandles of two elements up to relabeling.

### 4. GAME THEORY

**Problem 12** (2 points). In the following zero-sum game, Player 1 can play U or D while Player 2 can play L or R. Determine what each player chooses if both players play rationally.

	L	R
U	1	1
D	1	-2

**Problem 13** (2 points). If both players follow play maximin strategies, what will the outcome of the game be?

	L	R
U	1	2
D	1	-2

**Problem 14** (2 points). Consider the following game.

	L	R
U	2,1	0,0
D	0,0	1,2

- (a) Find the Pareto optimal outcomes.  
 (b) Find a mixed Nash equilibrium.

### 5. MISCELLANEOUS

**Problem 15** (3 points). Find the maximum value of the expression

$$H(x, y, z) = x \log(1/x) + y \log(1/y) + z \log(1/z)$$

subject to the constraints

$$x, y, z \geq 0, \quad x + y + z = 1.$$

**Problem 16.** Let  $A$  be a subset of the natural numbers  $\mathbb{N}$ . The **natural density** of  $A$  is defined as

$$\delta(A) := \lim_{N \rightarrow \infty} \frac{|A \cap [1, N]|}{N}$$

if the limit exists. For each of the following, decide if the natural density exists and if so, calculate it.

- (a) (1 point)  $A = \{2, 4, 6, 8, \dots\}$   
 (b) (1 point)  $A = \{n^2 : n \in \mathbb{N}\}$   
 (c) (2 points)  $A = \{n : n \text{ contains a 9 in its decimal expansion}\}$   
 (d) (2 points)  $A = \{n : \lfloor \log_2 n \rfloor \text{ is even}\}$   
 (e) (5 points)  $A = \{n : n \text{ is squarefree}\}$  (a number  $n$  is said to be **squarefree** if  $n$  is not divisible by  $p^2$  for any prime  $p$ )

**Problem 17** (3 points). Calculate  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ .

**Problem 18.** (2 points) Which is bigger,  $e^\pi$  or  $\pi^e$ ? Prove your answer (NO CALCULATORS) (obviously)

**Problem 19.** (5 points) Does the series

$$\sum_{p \text{ prime}} \frac{1}{p}$$

converge or diverge? Prove your answer.

**Problem 20.** (3 points) Calculate

$$\int_{-\infty}^{\infty} e^{-x^2} dx.$$

(Hint: polar coordinates)