MATH CIRCLE: ORDINARY GENERATING FUNCTIONS

LAMC OLYMPIAD GROUP

Definition (Polynomials, rational functions, power series)**.** Get ready for a long definition:

(1) A (complex) *polynomial* is a formal expression of the form

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

for some $n \geq 1$ and $a_0, \ldots, a_n \in \mathbf{C}$. If $a_n \neq 0$, we say that P is a polynomial of degree n.

(2) A *rational function* is a formal expression of the form

$$\frac{P(x)}{Q(x)},$$

where P and Q are polynomials.

(3) A *power series* is an infinite formal sum of the form

$$S(x) = \sum_{n\geq 0} a_nx^n = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots,$$

where $\{a_n\}_{n\geq 0}$ is a sequence of complex numbers. If we start with a sequence of complex numbers of interest and form the series above, we call $S(x)$ the *ordinary generating function* for the sequence $\{a_n\}_{n\geq 0}$; studying algebraic properties of $S(x)$ can help us study $\{a_n\}_{n\geq 0}$.

Remark. Above we defined polynomials, rational functions and power series as *formal expressions*, meaning that we can add them, multiply them **and even differentiate them** by the usual algebraic rules, but the indeterminate x is only a placeholder. Here's an example of a formal computation:

$$\left(1 + x + x^2 + x^3 + \cdots\right)(1 - x) = \begin{array}{l} 1 + x + x^2 + x^3 + \cdots \\ \quad - x - x^2 - x^3 - \cdots \end{array} = 1,$$

so we can say that, formally

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n\geq 0} x^n.$$

We've seen before that this is also true if x is an actual real number (not only a placeholder), *provided* that $-1 < x < 1$ (this also works for all complex numbers with $|x| < 1$). So we can regard polynomials, rational functions and power series as actual functions of complex numbers x, but then we need to be careful about dividing by 0 (for rational functions) and convergence (for series).

Problem 1. **(a)** Prove in two ways that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n\geq 0}(n+1)x^n$$

as formal expressions, and also as functions of x with $|x| < 1$.

(b) Show more generally that for any nonnegative integer k,

$$\frac{1}{(1-x)^{k+1}} = \sum_{n\geq 0} \binom{n+k}{k} x^n.$$

Problem 2. **(a)** Give a combinatorial proof of the binomial formula (for $n \in \mathbf{Z}_{\geq 0}$):

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

(b) Use part (a) to show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ and that $\sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0$.
(c) Use part (a) to show that when $|x| + |y| < 1$,

$$\frac{1}{1-x-y} = \sum_{n\geq k\geq 0} \binom{n}{k} x^k y^{n-k}$$

is the (two-variable) generating function for all binomial coefficients. How does this relate to Pascal's triangle?
(d) Plug in $y = 0$, $y = x$ and $y = -x$ in part (c) and see what you get. Compare this to part (b).

Problem 3. Show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

by expanding $(1+x)^n(1+x)^n = (1+x)^{2n}$ in two ways and identifying the coefficient of x^n.

Below are two key facts about polynomials. First, a recap of the fundamental theorem of algebra:

Theorem 1 (Polynomial factorization and Vieta)**.** *Any complex polynomial $P(x)$ of degree n factors as*

$$P(x) = a(x-x_1)(x-x_2)\cdots(x-x_n),$$

where $a \in \mathbf{C}$ is the leading coefficient of P, and $x_1, \ldots, x_n \in \mathbf{C}$ are all the zeros *of P (which may repeat). In this case, the coefficient of x^{n-k} in P is given by*

$$(-1)^k a \cdot \sum_{i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k}.$$

Now, if two rational functions are equal formally, then they are also equal wherever defined since they are literally the same expression. The great news is that the converse is also true:

Theorem 2 (Identity theorem)**.** *If two rational functions $F(x)$ and $G(x)$ give the same value for infinitely many choices of $x \in \mathbf{C}$, then they are equal wherever defined, and also equal formally. In particular, the factorization in Theorem 1 also holds as a formal identity.*

Example 3. Consider a second-degree polynomial, $P(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{C}$ and $a \neq 0$. You've probably seen that P has zeros at

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and by Theorem 1, we obtain the factorization

$$ax^2 + bx + c = a(x-x_1)(x-x_2),$$

which also holds formally by Theorem 2. This means that when we multiply the parentheses above we can identify the coefficients of x^0 and x^1 to obtain:

$$x_1 x_2 = \frac{c}{a}, \qquad x_1 + x_2 = \frac{-b}{a},$$

which you can also check directly (these are Vieta's formulas for $n = 2$).

Problem 4. **(a)** Show that $\frac{x}{1-x-x^2} = \sum_{n\geq 0} F_n x^n$ is the generating series for the Fibonacci numbers (defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$).
(b) Find numbers φ and ψ such that

$$1 - x - x^2 = (1 - \varphi x)(1 - \psi x).$$

Hint: let $y = x^{-1}$, multiply by y^2, and solve a quadratic equation; φ is called the golden ratio.
(c) Show that the following fraction decomposition holds:

$$\frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi x} - \frac{1}{1 - \psi x} \right).$$

(d) Conclude that, for all $n \geq 1$,

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

(e) Using Problem 2(c), show also that for all $n \geq 0$,

$$F_{n+1} = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n-m}{m}.$$

Problem 5. **(a)** For a positive integer n, let $p(n)$ be the number of *partitions* of n, i.e. the number of ways to write n as a sum $a_1 + \cdots + a_k$ with $1 \leq a_1 \leq \cdots \leq a_k$ arbitrary positive integers and $k \geq 0$. For example, $p(4) = 5$ since $1+1+1+1 = 1+1+2 = 1+3 = 2+2 = 4$. Show that the formal generating function of p has the form

$$\sum_{n\geq 0} p(n) x^n = \prod_{a\geq 1} \frac{1}{1 - x^a}.$$

(*b) Let $p_{\text{dist}}(n)$ be the number of partitions of n with *distinct parts a_i*, and $p_{\text{odd}}(n)$ be the number of partitions with *odd parts a_i*. Show that for all $n \geq 0$,

$$p_{\text{dist}}(n) = p_{\text{odd}}(n),$$

by comparing the two generating series.
Side-note: if you have time, look up Ramanujan's congruences for the partition function!

Problem *6. Define the following formal power series:

$$\operatorname{Exp}(x) := \sum_{n\geq 0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(a) Show that $\operatorname{Exp}(x + y) = \operatorname{Exp}(x) \cdot \operatorname{Exp}(y)$.
(b) What happens when you formally differentiate $\operatorname{Exp}(x)$ (term by term)?