# Lesson 8: Repetition

Nikita, Konstantin Myagkov

**Problem 1.**
Can you cover the quadrilateral from Figure 1 by 1024 quadrilaterals (which are similar and 32 times smaller) from Figure 2?

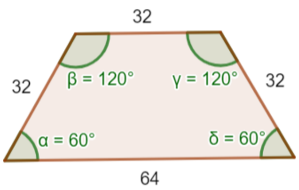



Figure 1

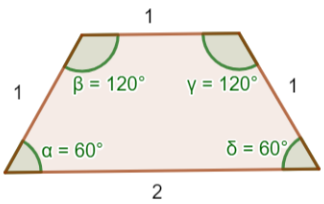
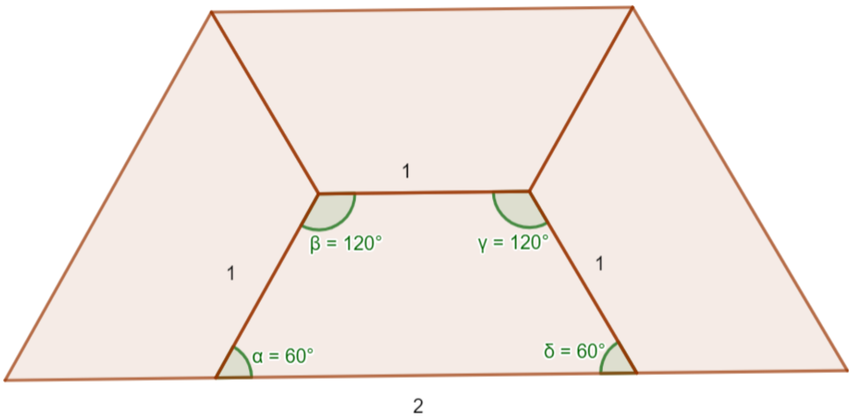



Figure 2

*Proof.* We can get a bigger similar quadrilateral out of 4 smaller ones like in the picture:




Then we build an even bigger quadrilateral out of 4 bigger ones. Repeat this trick 5 times to get a quadrilateral from Figure 1. □

**Problem 2.**
Prove the identity for all $n \geq 1$:

$$\frac{1^2+1}{2} + \frac{2^2+2}{2} + \frac{3^2+3}{2} + \cdots + \frac{n^2+n}{2} = \frac{n(n+1)(n+2)}{6}$$

*Proof.* Induction on $n$.

**Base:** If $n = 1$ this identity turns into $\frac{1^2+1}{2} = \frac{1\times2\times3}{6}$, which is true.

**Step:** Suppose it is true for $n = k$. Then

$$\frac{1^2+1}{2} + \frac{2^2+2}{2} + \frac{3^2+3}{2} + \cdots + \frac{(k+1)^2+(k+1)}{2}$$
$$= \left(\frac{1^2+1}{2} + \frac{2^2+2}{2} + \frac{3^2+3}{2} + \cdots + \frac{k^2+k}{2}\right) + \frac{(k+1)^2+(k+1)}{2}$$
$$\stackrel{IH}{=} \frac{k(k+1)(k+2)}{6} + \frac{(k+1)^2+(k+1)}{2}$$
$$= \frac{k(k+1)(k+2)+3(k+1)\Big((k+1)+1\Big)}{6}$$
$$= \frac{(k+1)(k+2)(k+3)}{6}.$$

So it is true for $k+1$ and (according to the principle of mathematical induction) for every $n$. □

**Problem 3.**
In a rectangle $m \times n$ some cells are marked red. It is known that in each of $m$ rows 10 cells are marked red and in each of $n$ columns 8 cells are marked red.

Prove that $m$ is divisible by 4 and $n$ is divisible by 5.

*Proof.* On the one hand, the number of red cells is $10 \times m$. On the other hand, the number of red cells is $8 \times n$. So

$$10m = 8n$$

or

$$5m = 4n.$$

Since 4 and 5 are coprime, $m$ is divisible by 4 and $n$ is divisible by 5.

□

**Problem 4.**
Triangle $ABC$ is acute. The bisector of angle $A$ and the perpendicular bisector of $BC$ intersect at point $D$. Show that the $ABDC$ is cyclic.

*Proof.* Compare with the problem 3 from the previous class:

"Triangle $ABC$ is acute. Prove that its circumscribed circle, bisector of angle $A$ and the perpendicular bisector of $BC$ all intersect at one point."

I copy it's solution here:

"Take the middle point $M$ of the shorter arc $BC$. Then both angle bisector and perpendicular bisector pass through it by 'equal angles stand on equal arcs or chords' theorem."

□

**Problem 5.**
How many ways are there to split the $2 \times 10$ board into dominoes?

*Proof.* Note that the upper left cell is either inside the horizontal or a vertical domino.

If it is inside the vertical domino, we need to split the remaining $2 \times 9$ board.

If it is inside the horizontal one, then below it we have one more horizontal domino, so we need to split the remaining $2 \times 8$ board.

Let's introduce $S(k)$ – the number of ways to split the $2 \times k$ board into dominoes, so $S(1) = 1$, $S(2) = 2$ and so on. We need to find $S(10)$. As we see, each way to split the $2 \times 10$ board comes from some splitting of either $2 \times 9$ or $2 \times 8$ board. So $S(10) = S(9) + S(8)$ board. Analogously we see that $S(k) = S(k-1) + S(k-2)$. So we can solve our problem from the end.

$$S(3) = S(2) + S(1) = 3;$$
$$S(4) = S(3) + S(2) = 5;$$
$$S(5) = S(4) + S(3) = 8;$$
$$S(6) = S(5) + S(4) = 13;$$
$$S(7) = S(6) + S(5) = 21;$$
$$S(8) = S(7) + S(6) = 34;$$
$$S(9) = S(8) + S(7) = 55;$$
$$S(10) = S(9) + S(8) = 89.$$

□

**Problem 6.**

**a)** Two squares of an $8 \times 8$ chess board have stones in them. With one move, it is possible to move one of the stones to an adjacent square. Is it possible to move the stones to a position symmetric to the initial position with respect to the central vertical line through the chessboard in exactly 2011 moves?

*Proof.* No, we can't. Consider the sum of the coordinates of both stones. If initially they have coordinates $(x_1, y_1)$ and $(x_2, y_2)$, that sum is $x_1 + y_1 + x_2 + y_2$. In he end it has to be $8 - x_1 + y_1 + 8 - x_2 + y_2$, which has the same parity. But with every move the sum changes by 1, so its parity changes. Thus after 2011 moves it has to have different parity, contradiction. □

**b)** Same question, but with 5 stones and central symmetry.

*Proof.* No. Same proof as in part a) works. □