The Cantor Set

Begin with the interval [0,1].

Remove the interval (1/3,2/3) (the "middle third").

What remains is the two intervals [0,1/3] and [2/3,1]. From each of these, remove the "middle third".

Continue ...

Is there anything left? "How much" was taken away?

The Devil's Staircase

For each number in the Cantor set, perform the following steps

1. Write the number in ternary (base 3), using only 0's and 2's.
2. Change all 2's to 1's.
3. Re-interpret the result in binary (base 2).

Draw a graph for the Cantor function by filling in the values for 1/3, 2/3, 1/9, 2/9, 7/9, 8/9, ...

For each number x which is not in the Cantor set, whereever the first 1 appears in its ternary expansion, chop off all digits past the 1, and replace the 1 with 0222222.... Then the result is in the Cantor set. Assign to x the same value as this number.

For example, the number $x = 0.22021201\ldots_3$ is replaced by $0.2202022222\ldots_3$, and assigned the value $0.110101111\ldots_2 = 0.11011_2 = 1/2 + 1/4 + 1/16 + 1/32 = 27/32$.

Summing Geometric Series

Consider the sum

$$S_n = 1 + \frac{1}{10} + \frac{1}{100} + \ldots + \frac{1}{10^n}.$$

Then

$$10S_n = 10 + 1 + \frac{1}{10} + \ldots + \frac{1}{10^{n-1}}.$$

Subtracting, we find that

$$9S_n = 10 - \frac{1}{10^n}$$
$$S_n = \frac{10}{9} - \frac{1}{9 \cdot 10^n}.$$

As the number n grows, the second term becomes smaller. So small, in fact that if we choose any given number $\varepsilon > 0$, there is an n large enough that $\frac{1}{9 \cdot 10^n} < \varepsilon$. (We will not give a rigorous proof of this fact. It's harder than you might think!)

As a result, we say that the sum S_n converges to 10/9.

Exercises

Find the sum of each of the following geometric series.

1) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$

2) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + ...$

3) $2 + \frac{4}{5} + \frac{8}{25} + \frac{16}{125} + ...$

Base Conversion

Our normal representation of a real number depends on powers of 10:

$$742.21 = 7 \cdot 10^2 + 4 \cdot 10^1 + 2 \cdot 10^0 + 2 \cdot 10^{-1} + 1 \cdot 10^{-2}.$$

A number can also be represented, for example, in base 8:

$$\begin{aligned} 321_8 &= 3 \cdot 8^2 + 2 \cdot 8 + 1 = 209 \\ 321.14_8 &= 3 \cdot 8^2 + 2 \cdot 8^1 + 1 \cdot 8^0 + 1 \cdot 8^{-1} + 4 \cdot 8^{-2} = 209.1875. \end{aligned}$$

... or in base 12:

$$\begin{aligned} 321_{12} &= 3 \cdot 12^2 + 2 \cdot 12 + 1 = 457 \\ 321.14_{12} &= 3 \cdot 12^2 + 2 \cdot 12^1 + 1 \cdot 12^0 + 1 \cdot 12^{-1} + 4 \cdot 12^{-2} = 457.\overline{1} \end{aligned}$$

Note that in base 12 we need symbols for each digit from 0 to 11, so we use a and b to stand for 10 and 11, respectively.

$$321.a4_{12} = 3 \cdot 12^2 + 2 \cdot 12^1 + 1 \cdot 12^0 + 10 \cdot 12^{-1} + 4 \cdot 12^{-2} = 457.86\overline{1}$$

Exercises

Perform each of the following conversions:

1. 25 into base 8

2. 25 into base 12

3. 25.125 into base 8

4. 32_8 into base 10

5. 32.1_8 into base 10