Definition. For a subset $X$ of $\mathbb{R}$, a point $x_0$ is said to be a \textit{limit point} iff there is a non-constant sequence in $X - \{x_0\}$ converging to $x_0$.

Let $D$ be a subset of $\mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function. If $x_0$ is a limit point of $D$, we say that $\lim_{x \to x_0} f(x)$ exists and is equal to $L$ iff for all non-constant sequences $a_n$ converging to $x$, the sequence $f(a_n)$ converges to $L$.

Definition. Let $X$ be a subset of $\mathbb{R}$. A function $f : X \to \mathbb{R}$ is said to be \textit{continuous} if for all points $x_0 \in I$, the limit of $f$ at $x_0$ exists, and

$$\lim_{x \to x_0} f(x) = f(x_0).$$

(*

Remark. The slogan is: continuous functions commute with taking limits. An alternative way to phrase this is that

$$a_n \to L \quad \Rightarrow \quad f(a_n) \to f(L),$$

the slogan here being: continuous functions preserve convergent sequences.

Examples. The functions $x^r, a^x, \sin x$ are continuous wherever defined.

Problem 1. Show that if $f, g$ are continuous functions, then $f + g$ and $f \cdot g$ are also continuous, wherever defined.

Problem 2. Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions, then so is their composite $g \circ f$.

Theorem. Let $I = [a, b]$ be a \textit{closed} interval, and $f : I \to \mathbb{R}$ be a continuous function. Then the image of $f$ (i.e. the set of values attained) is also a closed interval.

Corollary 1. The function $f$ attains both a maximum and a minimum.

Corollary 2. (Intermediate value property) If a real number $z$ is between $f(a)$ and $f(b)$, then there exists a point $c$ in $[a, b]$ so that $f(c) = z$.

Problem 3. Show that $f(x) := x^3 - 3x + 1$ has a real root (i.e., some $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$).

Problem 4. Let $n$ be a positive integer, and let $f : \{1, 2, \ldots, n\} \to \mathbb{Z}$ be a function such that $|f(k) - f(k + 1)| \leq 1$ whenever $1 \leq k < n$. Show that if $z$ is an integer between $f(1)$ and $f(n)$, then there exists an integer $c$ such that $f(c) = z$.

\textit{Hint: You can also solve this directly as a discrete problem, but here’s a nice trick: construct a continuous function $f : [1, n] \to \mathbb{R}$ which agrees with the original function on $\{1, \ldots, n\}$, then use the Intermediate Value Property. This will give you some $c \in [1, n]$ such that $f(c) = z$, and you can use the given assumptions to make $c$ be an integer (how?).}

Problem 5. Show that there are two antipodal points on Earth’s equator with the same altitude. (Assume altitude is a continuous function.)
**Definition.** Let $I$ be an interval. A function $f : I \to \mathbb{R}$ is **differentiable** at a point $x_0 \in (a, b)$ iff the limit
\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
\]
exists and is a real number. In this case, we denote its value by $f'(x_0)$, and we call it the “derivative of $f$ at $x_0$”. If $f$ is differentiable on all of $(a, b)$, then $f'$ becomes a new function.

**Problem 6.** Show that if $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$, then $f$ is also continuous at $x_0$ (so continuity is weaker than differentiability).

**Problem 7.** (Differentiation rules). Let $I$ be an interval.
(a) Show that if $f, g : I \to \mathbb{R}$ are differentiable, then
\[
(f + g)' = f' + g' \quad \text{and} \quad (f \cdot g)' = f'g + fg'.
\]
Moreover, if $g$ is nonzero on $I$ then
\[
\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.
\]
(b) Show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable, then
\[
(f \circ g)' = (f' \circ g) \cdot g',
\]
meaning that $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ for all $x \in I$.

**Problem 8.** All functions are understood on the intervals where they’re well-defined.
(a) Show that for $c \in \mathbb{R}$, the constant function $f(x) = c$ has $f' \equiv 0$.
(b) For a positive integer $n$, show that
\[
(x^n)' = nx^{n-1},
\]
and use this to conclude that
\[
(a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0)' = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1.
\]
(c) Show that
\[
(\sqrt{x})' = \frac{1}{2\sqrt{x}}.
\]
*Hint: Compute this directly from the definition of the derivative.*

**Remark.** Here’s a table with more derivatives:

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$x^r$ (for $r \neq 0$)</th>
<th>$e^x$</th>
<th>$a^x$ (for $a &gt; 0$)</th>
<th>$\sin x$</th>
<th>$\cos x$</th>
<th>$\ln x$ (for $x &gt; 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>$rx^{r-1}$</td>
<td>$e^x$</td>
<td>$a^x$</td>
<td>$\sin x$</td>
<td>$\cos x$</td>
<td>$\ln x$</td>
</tr>
</tbody>
</table>

It’s fun to check that, formally, these agree with the following infinite series for $e^x$, $\sin x$ and $\cos x$:
\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.
\]

**Problem 9.** Let $I$ be an interval. Show that if $f : I \to \mathbb{R}$ is differentiable and attains its maximum or minimum at $x_0$, then $f'(x_0) = 0$. (Since this works for any interval, it suffices to have a *local* maximum or minimum).
Problem 10. Let $a < b$ be real numbers.

(a) (Rolle’s Theorem). Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Show that if $f(a) = f(b)$, then
\[ \exists c \in (a, b) \quad \text{such that} \quad f'(c) = 0. \]

(b) (Lagrange’s Theorem, or Mean-Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Show that
\[ \exists c \in (a, b) \quad \text{such that} \quad f'(c) = \frac{f(b) - f(a)}{b - a}. \]

Problem 11. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$.

(a) Show that $f' = 0$ on all of $(a, b)$ if and only if $f$ is constant on $[a, b]$.

(b) Show that $f' \geq 0$ on all of $(a, b)$ if and only if $f$ is nondecreasing on $[a, b]$.

(c) Show that if $f' > 0$ on all of $(a, b)$, then $f$ is strictly increasing on $[a, b]$.

*Note that the function $x^3$ is strictly increasing on $[-1, 1]$, but $(x^3)'(0) = 0.*

Problem 12. Show that $x^4 - 4x + 4$ attains a minimal value on $\mathbb{R}$, and compute this minimum.

Problem 13. Let $I$ be an interval and $f, g : I \to \mathbb{R}$ be differentiable such that $f' = g'$ on $I$. Show that $f = g + c$ for some constant $c \in \mathbb{R}$. *Hint: Use Problem 11(a).*

(In other words, when we differentiate a function we don’t lose a lot of information about it; we can always recover it up to translation by a constant.)

Problem *14. (a) Show that $(x^e)' = x^e(\ln x + 1)$.

(b) Show that $x^e$ attains its global minimum (for $x > 0$) at $x = \frac{1}{e}$.

Problem 15. Show that the polynomial function $x \mapsto x^3 - 6x^2 + 12x - 18$ is strictly increasing. *Hint: Use Problem 11.*

Problem 16. Say $f : (0, \infty) \to \mathbb{R}$ is the function $f(x) = \sin (x^3) - 3 \cos (\sqrt{x})$. Using Problem 7(b) the table of derivatives on the previous page, compute $f''(x)$.

Problem 17. Say $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = \frac{x}{x^2 + 1}$.

(a) Using Problem 7(a), compute $f'(x)$.

(b) Using Problem 9, find all points where $f$ has a local minimum or a local maximum.

(c) Using Problem 11, determine where $f$ is increasing and where $f$ is decreasing (consider the intervals $(-\infty, -1], [-1, 1], [1, \infty)$ separately).