MATH CIRCLE: CONTINUITY AND DIFFERENTIABILITY

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Definition. For a subset X of $\mathbf{R}$, a point x_0 is said to be a *limit point* iff there is a non-constant sequence in $X - \{x_0\}$ converging to x_0.
Let D be a subset of $\mathbf{R}$, and let $f : D \to \mathbf{R}$ be a function. If x_0 is a limit point of D, we say that $\lim_{x \to x_0} f(x)$ exists and is equal to L iff for all non-constant sequences a_n converging to x, the sequence $f(a_n)$ converges to L.

Definition. Let X be a subset of $\mathbf{R}$. A function $f : X \to \mathbf{R}$ is said to be *continuous* if for all points $x_0 \in I$, the limit of f at x_0 exists, and

$$\lim_{x \to x_0} f(x) = f(x_0). \qquad (*)$$

Remark. The slogan is: continuous functions commute with taking limits. An alternative way to phrase this is that

$$a_n \to L \qquad \Longrightarrow \qquad f(a_n) \to f(L),$$

the slogan here being: continuous functions preserve convergent sequences.

Examples. The functions $x^r, a^x, \sin x$ are continuous wherever defined.

Problem 1. Show that if f, g are continuous functions, then $f + g$ and $f \cdot g$ are also continuous, wherever defined.

Problem 2. Show that if $f, g : \mathbf{R} \to \mathbf{R}$ are continuous functions, then so is their composite $g \circ f$.

Theorem. Let $I = [a, b]$ be a *closed* interval, and $f : I \to \mathbf{R}$ be a continuous function. Then the image of f (i.e. the set of values attained) is also a closed interval.

Corollary 1. The function f attains both a maximum and a minimum.
Corollary 2. (Intermediate value property) If a real number z is between $f(a)$ and $f(b)$, then there exists a point c in $[a, b]$ so that $f(c) = z$.

Problem 3. Show that $f(x) := x^3 - 3x + 1$ has a real root (i.e., some $x_0 \in \mathbf{R}$ such that $f(x_0) = 0$).

Problem 4. Let n be a positive integer, and let $f : \{1, 2, \dots, n\} \to \mathbf{Z}$ be a function such that $|f(k) - f(k+1)| \leq 1$ whenever $1 \leq k < n$. Show that if z is an integer between $f(1)$ and $f(n)$, then there exists an integer c such that $f(c) = z$.
Hint: You can also solve this directly as a discrete problem, but here's a nice trick: construct a continuous function $f : [1, n] \to \mathbf{R}$ which agrees with the original function on $\{1, \dots, n\}$, then use the Intermediate Value Property. This will give you some $c \in [1, n]$ such that $f(c) = z$, and you can use the given assumptions to make c be an integer (how?).

Problem 5. Show that there are two antipodal points on Earth's equator with the same altitude. (Assume altitude is a continuous function.)

Definition. Let I be an interval. A function $f : I \to \mathbf{R}$ is *differentiable* at a point $x_0 \in (a, b)$ iff the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is a real number. In this case, we denote its value by $f'(x_0)$, and we call it the "derivative of f at x_0". If f is differentiable on all of (a, b), then f' becomes a new function.

Problem 6. Show that if $f : I \to \mathbf{R}$ is differentiable at $x_0 \in I$, then f is also continuous at x_0 (so continuity is weaker then differentiability).

Problem 7. (Differentiation rules). Let I be an interval.
(a) Show that if $f, g : I \to \mathbf{R}$ are differentiable, then

$$(f + g)' = f' + g' \qquad \text{and} \qquad (f \cdot g)' = f'g + fg'.$$

Moreover, if g is nonzero on I then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

(b) Show that if $f : \mathbf{R} \to \mathbf{R}$ and $g : I \to \mathbf{R}$ are differentiable, then

$$(f \circ g)' = (f' \circ g) \cdot g',$$

meaning that $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ for all $x \in I$.

Problem 8. All functions are understood on the intervals where they're well-defined.
(a) Show that for $c \in \mathbf{R}$, the constant function $f(x) = c$ has $f' \equiv 0$.
(b) For a positive integer n, show that

$$(x^n)' = nx^{n-1},$$

and use this to conclude that

$$\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\right)' = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1.$$

(c) Show that

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}.$$

Hint: Compute this directly from the definition of the derivative.

Remark. Here's a table with more derivatives:

$f(x)$	x^r (for $r \neq 0$)	e^x	a^x (for $a > 0$)	$\sin x$	$\cos x$	$\ln x$ (for $x > 0$)
$f'(x)$	rx^{r-1}	e^x	$(\ln a)a^x$	$\cos x$	$-\sin x$	$1/x$

It's fun to check that, formally, these agree with the following infinite series for e^x, $\sin x$ and $\cos x$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Problem 9. Let I be an interval. Show that if $f : I \to \mathbf{R}$ is differentiable and attains its maximum or minimum at x_0, then $f'(x_0) = 0$. (Since this works for any interval, it suffices to have a *local* maximum or mininimum).

Problem 10. Let $a < b$ be real numbers.
(a) (Rolle's Theorem). Suppose $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Show that if $f(a) = f(b)$, then

$$\exists c \in (a, b) \qquad \text{such that} \quad f'(c) = 0.$$

(b) (Lagrange's Theorem, or Mean-Value Theorem). Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that

$$\exists c \in (a, b) \qquad \text{such that} \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 11. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b).
(a) Show that $f' = 0$ on all of (a, b) if and only if f is constant on $[a, b]$.
(b) Show that $f' \geq 0$ on all of (a, b) if and only if f is nondecreasing on $[a, b]$.
(c) Show that if $f' > 0$ on all of (a, b), then f is *strictly* increasing on $[a, b]$.
Note that the function x^3 is strictly increasing on $[-1, 1]$, but $(x^3)'(0) = 0$.

Problem 12. Show that $x^4 - 4x + 4$ attains a minimal value on $\mathbf{R}$, and compute this minimum.

Problem 13. Let I be an interval and $f, g : I \to \mathbf{R}$ be differentiable such that $f' = g'$ on I. Show that $f = g + c$ for some constant $c \in \mathbf{R}$. *Hint: Use Problem 11(a).*
(In other words, when we differentiate a function we don't lose a lot of information about it; we can always recover it up to translation by a constant.)

Problem *14. **(a)** Show that $(x^x)' = x^x(\ln x + 1)$.
(b) Show that x^x attains its global minimum (for $x > 0$) at $x = \frac{1}{e}$.

Problem 15. Show that the polynomial function $x \mapsto x^3 - 6x^2 + 12x - 18$ is strictly increasing.
Hint: Use Problem 11.

Problem 16. Say $f : (0, \infty) \to \mathbb{R}$ is the function $f(x) = \sin\left(x^3\right) - 3\cos\left(\sqrt{x}\right)$. Using Problem 7(b) the table of derivatives on the previous page, compute $f'(x)$.

Problem 17. Say $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = \frac{x}{x^2+1}$.
(a) Using Problem 7(a), compute $f'(x)$.
(b) Using Problem 9, find all points where f has a local minimum or a local maximum.
(c) Using Problem 11, determine where f is increasing and where f is decreasing (consider the intervals $(-\infty, -1]$, $[-1, 1]$, $[1, \infty)$ separately).