GAMES AND STRATEGY

1. TAKE AWAY GAMES

Below you will find 5 different Take Away Games, each of which you may have played last year. Play each game with your partner. Find the winning strategy. As a hint, start by analyzing the smallest games. Then, use your results to determine strategies for larger games. This is sometimes called, "backwards" or "inductive" reasoning. The player who moves first will be called Player I and the player who moves second will be called Player II. For each game, begin with the setup. Then, alternate turns between the two players each making a single move per turn. The player who reaches the goal first, wins!
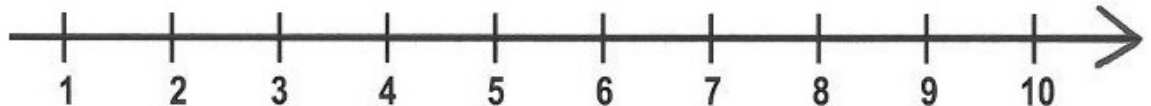
(1) **Setup:** One pile of n chips.
Move: Remove exactly one chip from the pile.
Goal: Remove the last chip.
  (a) For which values of n does Player I win?

  (b) For which values of n does Player II win?

  (c) Above each number on the number line below, mark which player will win the game beginning with that number of chips.

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(2) **Setup:** One pile of n chips.
  Move: Remove either one or two chips from the pile.
  Goal: Remove the last chip.

  (a) For which values of n does Player I win?

  (b) For which values of n does Player II win?

  (c) Above each number on the number line below, mark which player will win the game beginning with that number of chips.
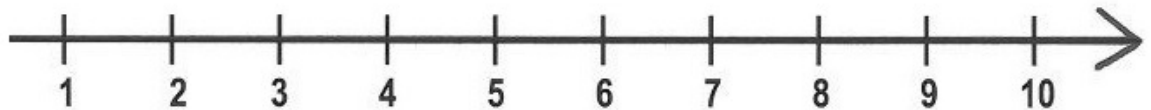
(3) **Setup:** One pile of n chips.
  Move: Remove either one, two, or three chips from the pile.
  Goal: Remove the last chip.

(a) For which values of n does Player I win?

(b) For which values of n does Player II win?

(c) Above each number on the number line below, mark which player will win the game beginning with that number of chips.
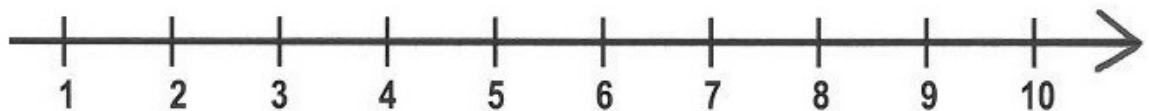
LAMC handout   4

(4) **Setup:** One pile of n chips.
    Move: Remove either one or three chips from the pile.
    Goal: Remove the last chip.

(a) For which values of n does Player I win?

(b) For which values of n does Player II win?

(c) Above each number on the number line below, mark which player will win
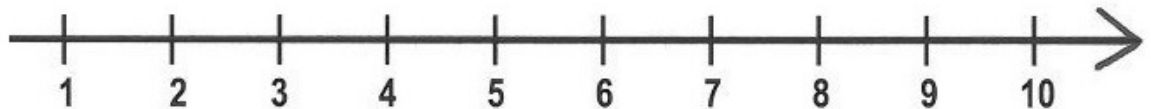   the game beginning with that number of chips.

(5) **Setup:** One pile of n chips.
 Move: Remove either two or three chips from the pile.
 Goal: Remove the last chip.

    (a) For which values of n does Player I win?

    (b) For which values of n does Player II win?

    (c) Above each number on the number line below, mark which player will win the game beginning with that number of chips.
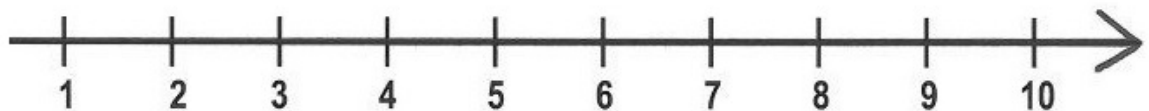
2. NIM

Nim is a more general version of the subtraction game. We shall use the same method as before, analyzing small games first, to determine a winning strategy for Nim. First we will examine the smallest version of Nim where there are only two piles.

2.1. **Nim with Two Equal Piles.**

(1) **Setup:** Two piles of n chips each.
Move: Remove some positive number of chips (possibly the entire pile) from a single pile.
Goal: Remove the last chip.

    (a) Begin by examining small games. For each of the following values of n, determine which player has a winning strategy, Player I or Player II.
        (i) $n = 1$

        (ii) $n = 2$

        (iii) $n = 3$

    (b) Describe the pattern you see above. For what values of n can Player II win?

    (c) In your own words, describe the general strategy for Nim with two equal piles. Be sure to indicate whether it is best to go first or second.

2.2. **Nim with Two Unequal Piles.**

(1) **Setup:** One pile of m chips and one pile of n chips, where $m \neq n$.
Move: Remove some positive number of chips (possibly the entire pile) from a single pile.
Goal: Remove the last chip.

(a) Begin by examining small games. For each of the following values of m and n, determine which player has a winning strategy, Player I or Player II. Keep in mind the winning strategy for Nim with two equal piles.

(i) $m = 1, n = 2$

(ii) $m = 2, n = 3$

(iii) $m = 3, n = 5$

(b) Describe the pattern you see above. For what values of m and n can Player II win?

(c) On your turn, if the two piles are equal, can you make a valid move such that afterwards, the piles are still equal? If so, what move? If not, why not?

(d) On your turn, suppose the two piles are unequal with a chips in one pile and b chips in the other, where $a > b$. Can you make a valid move such that afterwards, the piles are equal? If so, what move? If not, why not?

(e) In your own words, describe the general strategy for Nim with two piles (equal or unequal). Be sure to indicate when it is best to go first and when it is best to go second.

2.3. **Nim with Three Equal Piles.**

(1) **Setup:** Three piles of n chips.
Move: Remove some positive number of chips (possibly the entire pile) from a single pile.
Goal: Remove the last chip.

(a) Begin by examining small games. For each of the following values of n, determine which player has a winning strategy, Player I or Player II. Keep in mind what the winning strategy was for Nim with two piles.

(i) $n = 2$

(ii) $n = 3$

(iii) $n = 5$

(b) Describe the pattern you see above. For what values of n can Player I win?

(c) In your own words, describe the general strategy for Nim with three equal piles. Be sure to indicate whether it is best to go first or second.

2.4. **Nim with Two Equal Piles, One Unequal Pile.**

(1) **Setup:** Two piles of n chips, one pile of m chips, with $m \neq n$.
Move: Remove some positive number of chips (possibly the entire pile) from a single pile.
Goal: Remove the last chip.

  (a) Find the winning strategy for this version of Nim.

  (b) How does this game differ from Nim with three equal piles?

3. NIM-LIKE GAMES

As it turns out, there are many games similar to Nim. We can use the same strategy of backwards reasoning, analyzing small games first and using the results to analyze large games, in order to find winning strategies for these games.

3.1. **Empty and Divide.**

(1) **Setup:** Two boxes, Box A with m chips and Box B with n chips. We will write this position as (m, n). **Move:** Empty out one of the boxes. Split the contents of the other box between the two boxes any way you like (you do not have to divide evenly). You must place **at least one chip** in each box. **Goal:** Leave one chip in each box (leaving your opponent with no valid move). That is, reach position $(1, 1)$.

    (a) Play a game of Empty and Divide (E&D) with $m = 2, n = 1$.
        (i) Who wins, Player I or Player II?

        (ii) What is the optimal first move?

        (iii) We say a player has **no freedom of choice** if every option available to them results in the same outcome. Does Player I have freedom of choice for these values of m and n? Why or why not?

        (iv) If Box A has only 1 chip in it, can Box B be emptied? Why or why not?

(b) Play a game of E&D with starting position $m = 2, n = 2$.

   (i) Who wins, Player I or Player II?

   (ii) What is the optimal first move?

   (iii) Describe how Player I still has no freedom of choice, even though he can choose Box A or Box B to empty.

(c) Play a game of E&D with starting position $m = 2, n = 3$.
   (i) Who wins, Player I or Player II?

   (ii) What is the optimal first move?

   (iii) This time, Player I has two possible moves. Is the other move also a winning move? Why or why not?

(d) Play a game of E&D with starting position $m = 2, n = 100$.

   (i) Who wins, Player I or Player II?

   (ii) What is the optimal first move?

   (iii) For any starting position where $m = 2$ or $n = 2$, describe the winning strategy for Player I.

(e) Play a game of E&D with starting position $m = 3, n = 3$

   (i) Who wins, Player I or Player II?

   (ii) Does Player I have freedom of choice? Why or why not?

   (iii) Describe the winning strategy for this game.

(f) Continue playing larger games of E&D. Record the pairs of m and n for which Player I has a winning strategy and the pairs for which Player II has a winning strategy. Once you have done enough experimenting to see a pattern, draw a conclusion.

   (i) Player I Wins: $(2, 1)$,

   (ii) Player II Wins: $(1, 1)$,

   (iii) What is your conclusion? What must be true of m and n for Player II to have a winning strategy?

(g) Let's investigate to determine why this pattern exists.
   (i) Compute the following sums:
      (A) 3+4=

      (B) 7+11=

      (C) 212+10=

   (ii) Suppose n is an odd natural number. If $n = a + b$ for some natural numbers a and b, what can we say about the **parity** (odd-ness or even-ness) of a and b?

(iii) Suppose it is your turn and both boxes have an odd number of chips. Is it possible to make a valid move such that afterward, both boxes still have an odd number of chips? If so, how? If not, why not?

(iv) Let $2m$ be an even natural number. Can you find two odd numbers a and b such that $2m = a + b$? What are they? (Remember: this should work for **any** natural number m, even $m = 1$).

(v) Suppose on your turn the position is $(2m, n)$. Describe the strategy you can use to win the game.

3.2. **Chomp.**

(1) **Setup:** A rectangular chocolate bar, with a poisonous bottom left square. On the grid, call this bottom left position $(1, 1)$
Move: Select a square on the bar. Break off and eat all squares above and to the right of the chosen square. In the example below, if Player I chooses square $(17, 10)$, he will chomp the entire yellow rectangle.
Goal: Do NOT eat the bottom left square. That is, leave your opponent with only the poisoned square.
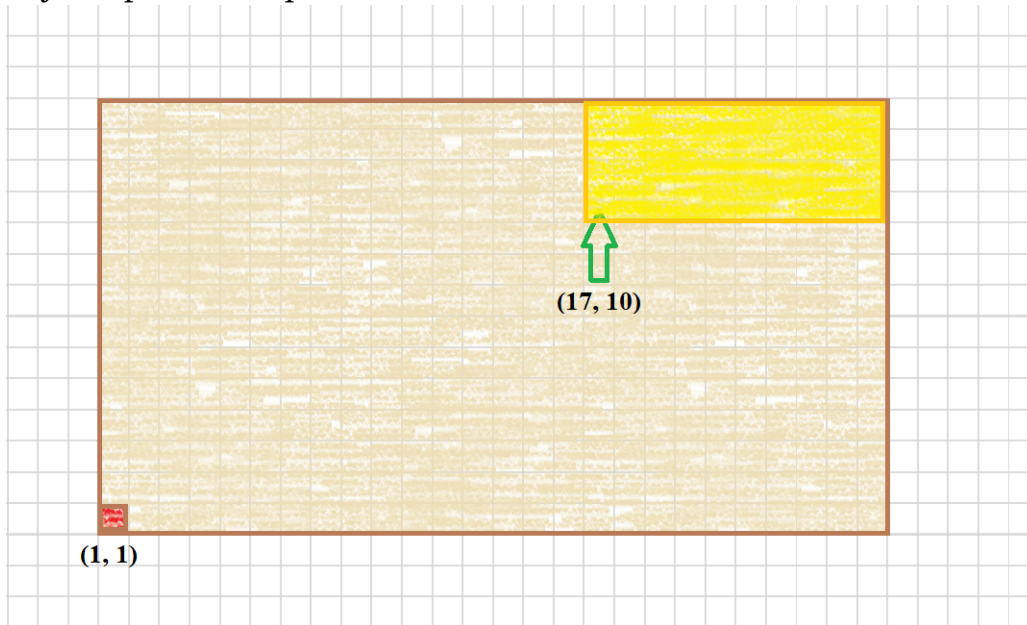
(a) On a piece of graph paper, draw a 2 by 2 Chomp board to play with your partner.

(i) How many possible first moves does Player I have? Do not count choosing square $(1, 1)$ as a move, since it is essentially surrender.

(ii) Of these, which move(s) result in a win?

(iii) Is there a winning strategy for either player? If so, what is it?

(b) Play a new game of Chomp, this time on a 3 by 3 board.

(i) Use your results from part (a) to find a winning strategy for this new game. Describe any similarity between the 2 by 2 game and the 3 by 3 game.

(ii) What is the winning strategy for the 4 by 4 game of Chomp?

(iii) What is the winning strategy for the n by n game of Chomp?

(iv) (Challenge) What is the relationship between Chomp on a square board and Nim with two piles?

(c) Other than square boards, there is another type of rectangular board for which we can describe the winning strategy. Play a game of Chomp, this time on a board with height 2 and length 3.

    (i) Use your results from part (a) to find a winning strategy for this new game. Describe any similarity between the 2 by 2 game and the 2 by 3 game.

    (ii) What is the winning strategy for the 2 by 4 game of Chomp?

    (iii) What is the winning strategy for the 2 by n game of Chomp?

(d) For the general game of chomp on an m by n board, we cannot write down an explicit winning strategy for either player. However, mathematicians are clever, and can still prove that Player II does **not** have a winning strategy by using what is called a **strategy-stealing** argument.

    (i) Suppose Player II **does** have a winning strategy. That is, no matter what first move Player I makes, Player II has a follow-up move that will guarantee him a win. Suppose for the first move, Player I bites only the top right square. Can Player II win?

    (ii) For the second move, Player II follows his winning strategy and selects square (m, n), guaranteeing him a win. If Player I chose square (m, n) on the first turn, would the board look the same or different?

    (iii) Explain in your own words why Player II having a winning strategy would imply that Player I has a winning strategy. Why does this mean Player II cannot have a winning strategy?

Fun fact, the winning strategy for Chomp on a 3 by n board was solved in 2002 by a high school senior!

3.3. **Dynamic Subtraction.**

(1) **Setup:** One pile of n chips.
Move: Remove at least one chip from the pile, but **no more than were taken on the previous turn.** On the first turn, Player I must take less than the entire pile.
Goal: Remove the last chip.

(a) In this game, we will identify a game position by two numbers: the number of chips remaining in the pile and the number of chips that were removed last turn. So, the game position where 3 balls remain and 1 ball was removed on the last turn will be written $(3, 1)$.

(i) If you are in the position $(3, 1)$, will you win or lose? Why?

(ii) Suppose you are in the position $(n, 1)$. For which values of n will you win, and what is the optimal move?

(iii) For which values of n will you lose?

(b) Suppose you are in position $(n, 2)$. That is, n chips remain and on your opponent's last turn they removed 2 chips.

    (i) If n is odd, what is the optimal move? Keep in mind your results from part (a).

    (ii) Suppose n is not odd. For which values of n will you win, and what is the optimal move?

    (iii) For which values of n will you lose?

(c) What is an optimal move for Player I if the game begins with 44 balls in the pile?

(d) For what starting positions will Player II win?