Sticks and Stones
Version 1.0
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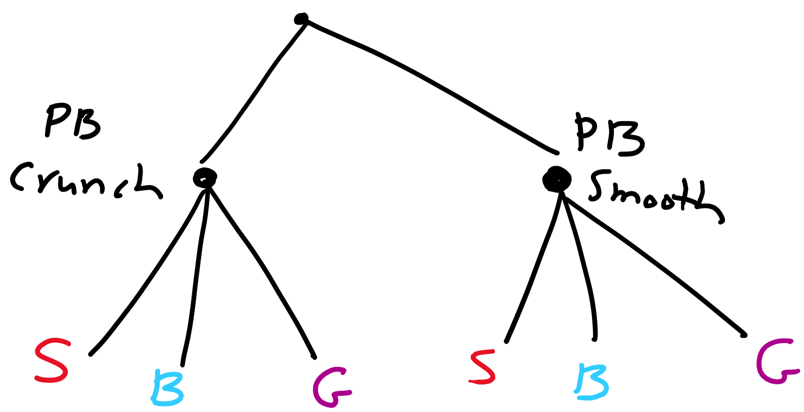
This week, we consider a special type of counting problem. Before doing so, however, it is helpful to review a few counting strategies that you probably have already seen in school or in Math Circle.

REVIEW

1. I am making peanut butter and jelly sandwiches. I have two types of peanut butter available to me: crunchy and smooth. I have three types of jelly: strawberry, blueberry, and grape. How many different sandwich combinations can I make, assuming I will put one type of peanut butter and one type of jam on any sandwich?

1a. Start by drawing a chart that shows the possibilities.



2 PB × 3 JAMS = 6 Possible Sandwiches

1b. Now, write down the math that takes you to the same answer. Do you see how the math and the chart relate?

$$\underbrace{2}_{\text{PB types}} \times \underbrace{3}_{\text{Jelly types}} = \underbrace{6}_{\text{Sandwich types}}$$

Yum! :)

2. Three of my friends (A,B,C) are getting together to play SpaceShip Commander. One of us can be Captain, one can be Science Officer, and one can be Navigator. How many different options do we have for today's game?

2a. Start by making the list by hand.

	Capt	Sci	Nav	
Captain A →	A	B	C	} 2
	A	C	B	
Captain B →	B	A	C	} 2
	B	C	A	
Captain C →	C	A	B	} 2
	C	B	A	

2b. Now, write down the math that takes you to the same answer. Again, do you see how the math and the chart relate?

$$\underset{\text{CAPTAINS}}{\underline{3}} \times \underset{\text{SCI}}{\underline{2}} \times \underset{\text{NAV}}{\underline{1}}$$

No choice: 1 friend left

After choosing the CAPTAIN I have only 2 options left

I have 3 options: A, B, C

3. Three of my friends gather together to play. We decide to play the 2-player game, Battleship. That means two friends will play, while one will just watch. How many possible pairings are there?

3a. Write out the list by hand. Be sure to eliminate duplicates.

AB BC
AC

BA = AB
CA = AC
CB = BC

3b. Now write down the math that takes you there. Again, watch out for duplicates.

$$\underline{3} \times \underline{2} = 6 \underline{\div 2} = 3$$

first player second player Get rid of duplicates

3c. Re-imagine the problem but focused on "leaving one person out" instead of focusing on the players. That is, imagine I had asked, "How many different people can be left out, while everyone else plays a 2-player game?" Easier, no?

A or B or C = 3

Wow!

STICKS AND STONES // EXAMPLE

I have 8 stones and I want to divide them into 3 piles, where each pile must have at least one stone. So I might have piles of 2/2/4, or 4/3/1, or 3/3/2, and so on. How many options do I have in total?

Start by drawing your 8 stones in a line.

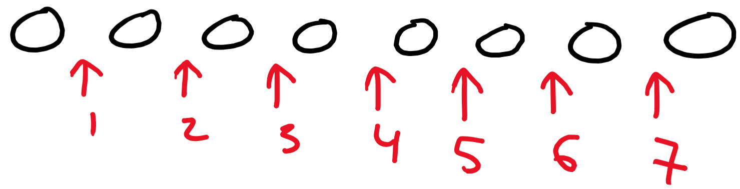
O O O O O O O O

Now, notice that we can make three piles by placing two sticks. For example:

O O | O O O O / O O

2 4 2

To find all the possibilities, all we have to do is realize that our first stick can be placed in any of 7 different locations, our second stick can be placed in any of 6 different locations, and so there are 42 possible stick placements. Eliminating duplicates, however, cuts the number in half.



$$\underset{\text{Stick 1 options}}{\underline{7}} \times \underset{\text{Stick 2 options}}{\underline{6}} = 42 \div 2 = 21$$

STICKS AND STONES // QUESTIONS

1. I add three numbers together to get 12. Assume that all three have to be positive, non-zero numbers, such as 1, 2, 3 and so on. How many different groups of numbers will work? Start by listing out a few examples, and then use sticks-and-stones to count them all.

$5+5+2$ or $3+4+5$

* 12 stones
* 11 spaces
* 2 sticks

$$11 \times 10 = 110 \div 2 = 55$$

2. I add three numbers together to get 9. But this time allow any of the numbers to be zero. So, for example, 3+0+6 is a permissible answer. How many different groups of numbers will work? Again, start by listing some examples, then use sticks-and-stones to count them all.

$6+2+1$
$7+1+1$
$5+0+4$

How do we solve??
Do a related problem that you know how to do.

Add 1 to each number.
Now, you know how to solve.
We just eliminated the zeros.

$x+y+z=12$ +3

+1 +1 +1

* 12 stones
* 2 sticks
* 11 spaces

$11 \times 10 = 110 \div 2 = 55$

Think of it this way:

○○○|○|○○○○○○○○

$3 \quad 1 \quad 8 \quad = 12$ ← No zeros allowed

$2 \quad 0 \quad 7 \quad = 9$ ← Allows ZERO

3. I have ten coins in my pocket. They might be nickels, dimes or quarters. How many possible groupings might I have?

Add +3 so we can have ZEROs.

0000|00000|0000 13 stones

12 spaces
2 sticks

$12 \times 11 = 132 \div 2 = 66$

4. Three numbers add to 11. But the numbers can only be 1,2,3 through 8, but not 9, 10, 11 or 12. How many options this time?

Do the normal problem first.

- 11 stones
- 10 spaces
- 2 sticks

$10 \times 9 = 90 \div 2 = 45$

Now... what to eliminate?

9 + 1 + 1
1 + 9 + 1
1 + 1 + 9

← these are the only extras, there is no 10 + ... or 11 + ...

5. How many ways are there to roll a sum of 7 with three standard six-sided dice?

O O O O O O O

6 spaces

2 sticks

No zeros.

$6 \times 5 = 30 \div 2 = \underline{\underline{15}}$